Optimal sharing in social dilemmas

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Abstract

Public goods games are frequently used to model strategic aspects of social dilemmas and to understand the evolution of cooperative behaviour among members of a group. While providing a baseline case, a (local) public goods model implies an equal sharing of returns. This appears an unsatisfying modelling choice in contexts where contributors are heterogeneous and returns can be divided freely. Furthermore, it is intrinsically linked to the negative effect of inequality on cooperation, which is observed both theoretically and experimentally. To better understand the link between inequality and cooperation when returns can be shared flexibly, we characterise sharing behaviour that maximises contributions in an infinitely repeated voluntary contribution game, where players differ in both their endowments as well as the productivities of their contributions. In sharp contrast to egalitarian sharing, we find that endowment inequality makes cooperation easier to sustain when returns can be shared unequally. Maybe surprisingly, this qualitative relation between endowment inequality and cooperation is independent of players' productivities. We derive a unique sharing rule as a function of productivities and endowments that is weakly superior to all other sharing rules. This rule generically departs from both equal as well as proportional sharing. If inequality is high, for example, individuals with the highest endowment need to be compensated more in absolute terms, but their relative share may be significantly less than their proportional contribution. Our analytical findings are qualitatively supported by numerical simulations of simple evolutionary learning dynamics.

1 Introduction

Social and economic life is rich in examples of *social dilemmas*; situations where cooperation among several individuals or entities is required to successfully complete a task but there is a tension between the welfare maximising outcome and individual incentives. Facilitating effective teamwork is a key objective for many companies, cooperation between different countries is required to tackle complex problems like climate change or the decline in biodiversity, collaboration among different researchers is essential for scientific progress, and success in many sporting competitions relies not just on fielding a team with the strongest individuals but one that works in unison. In such joint tasks, equal division seems an ideal scenario: the division rule is simple, no information about inputs is required, and allocations are envy free. In the case of hunter-gatherers, equal prey sharing guarantees the same amount of food and, hence, survival chance to everyone. However, Nature is rich in examples where goods are not shared equally but according to different rules: public-goods producers in cancer cells develop more efficient mechanisms of the good consumption (Li and Thirumalai, 2019), successful hunters gain a larger share of the prey in some hunter-gatherer communities (Gurven, 2004), and not all members of a sports team are remunerated equally. Yet, in many evolutionary or repeated interaction models, it is explicitly or implicitly assumed that returns are divided equally among members. For instance, modelling these interactions as a (local) public goods game necessarily corresponds to an equal sharing of the good, as all participants receive the same return. As is shown in many experiments, such egalitarian sharing can lead to an increased incentive to free ride and a breakdown of cooperation (Dal Bó and Fréchette, 2018).

At the root of this issue lies the inequality or 'heterogeneity' of contributors. Equal sharing of rewards seems innocuous if all collaborators are identical. Yet, inequality is a general feature of human societies. Team members don't usually have identical skill sets, experience, or even opportunity cost, countries vary in their economic capabilities, and athletes not only differ in skills, but also their bargaining power. Compensating individuals equally in spite of differences in the magnitude and quality of contributions seems intuitively unfair and studies have systematically confirmed the tension between inequality and cooperation. For instance, Hauser et al. (2019) recently highlighted the decline in cooperation that comes with endowment inequality in voluntary contribution games; both in a classic repeated game as well as an evolutionary setting. In social dilemmas with equal sharing, cooperation becomes harder to sustain the more unequal players are. High inequality in wealth or productivity can render cooperation impossible, even with perfectly patient individuals. When inequality becomes sufficiently extreme, the share obtained under equal sharing can no longer sufficiently compensate the largest contributors. Nevertheless, despite obvious inequalities, cooperation is widely observed in practice. It thus appears intuitive that relaxing the equal-sharing constraint might somewhat alleviate the issues caused by inequality. However, it is not obvious how inequality and unequal sharing interact. For instance, if rewards can be shared freely, how does inequality affect the ability of individuals to cooperate? How should rewards be shared to maximise cooperation? The focus of this study thus lies on analysing how heterogeneity of individuals, and the possibility to share rewards unequally, jointly affect cooperation in social dilemmas.

We study the effects of heterogeneity in a simple n-player, infinitely repeated game. In every period, players simultaneously decide how much of their endowment they contribute to a publicly produced good. The welfare maximum is reached if all players contribute their entire endowments. Players' individual payoffs consist of their retained endowment as well as their share in the jointly produced good. The relative share each player obtains is fixed, publicly known, and independent of inputs. This implies that sharing rules are committed but individual contributions are not verifiable. Players can differ in their endowment and the productivity of their contributions. In an extension, differences in outside options are also considered. Consistent with many of the described applications, players do not have access to explicit punishment devices but can enforce cooperation only through potential (future) reductions in their own contributions. To characterise the limits of cooperation, we focus on what is known as a 'grim-trigger strategy', where deviations are punished permanently and as harshly as individually rational. In the game considered here, this is without loss of generality. A schematic representation of the model can be found in Figure 1. Our main goal is to characterise the (optimal) sharing rule that can sustain full cooperation and analyse its relation with inequality. We further examine the evolution of cooperation under introspection dynamics from Hauser et al. (2019) and compare this qualitatively to the derived rule.

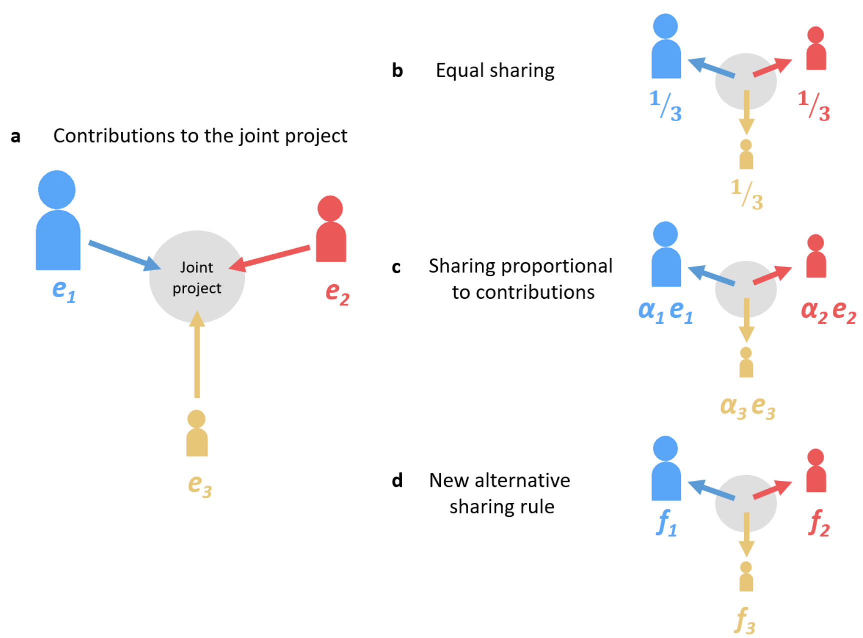

Figure 1: (a) A schematic representation of the setup. Heterogeneous players (in this case, unequal in their endowments) contribute to the joint project. (b) Under egalitarian sharing, each individual receives $1/3$ of the project output independent of their contribution. (c) Under proportional sharing rules, individuals with larger contributions receive higher shares. (d) However, proportional sharing might result in cooperation unravelling due to individual incentives. Hence, we want to identify a sharing rule that sustains cooperation at equilibrium.

In line with the previously outlined, intuitive reasoning, we show that unequal sharing generally makes cooperation easier to sustain, in the sense that it allows for cooperation in equilibrium for a larger range of discount factors. However, maybe surprisingly and to the contrary of the conclusions drawn under equal sharing, we show that cooperation becomes easier to sustain when endowments are more unequal. For instance, even if all contributors are equally productive, endowment inequality facilitates cooperation. This implies that if individuals can split their endowment across two identical projects, the intuitive approach of all individuals splitting their contributions equally across projects is not the approach most conducive to cooperation. It is also shown that, unlike under equal sharing, the higher the inequality in (potential) maximal contributions, the lower the required discount factor that sustains full cooperation. Furthermore, this effect is not linked to productivities. As in the equal sharing case, cooperation benefits if inequality in endowments and productivities are aligned, meaning that the richest players have the highest productivity. However, this is not a necessary condition under unequal sharing. Inequality in endowments can benefit cooperation even if the potentially largest contributor is the least productive. We characterise the conditions an optimal sharing rule needs to fulfil. For a generic set of parameters, such a rule is not unique, which leaves room for bargaining among contributors. However, we derive a sharing rule that sustains full cooperation for a lower discount factor than any other rule. It is in this sense (weakly) superior to all other sharing rules. Furthermore, at the lowest discount factor that sustain cooperation, this is the only sharing rule that sustains cooperation. We furthermore present a simple way to jointly derive this optimal rule and the corresponding minimum discount factor.

One striking implication of our results is that individuals with the highest endowment need to be compensated proportionally less than individuals with smaller contributions. The optimal sharing rule thus also deviates from proportional sharing rules where players are compensated according to their relative contributions (Moulin, 2002). Furthermore, higher productivity decreases the minimum share a player needs to be guaranteed to contribute. These findings are qualitatively robust to allowing for players to have heterogeneous outside options. That is, rather than contributing to the joint project, players can invest in their personal project with different rates of return.

The paper is structured as follows: We first provide a review of the related literature. Section 2 then formally outlines the model. In Section 3, we provide all analytical results. First, we characterise the properties any sharing rule needs to satisfy to facilitate (full) cooperation. We derive a unique optimal sharing rule as a function of endowments and productivities that can sustain cooperation for the widest range of discount factors. We then analyse the effect of endowment inequality on the derived sharing rule and the sustainability of cooperation in general. We contrast this with equal and proportional sharing rules. We then extend the analysis by allowing for heterogeneity in players' outside options. In addition to the analytical results, Section 4 introduces evolutionary dynamics and presents the corresponding numerical simulation. We conclude with the discussion of our results in Section 5.

1.1 Related literature

The evolution of human societies from relatively egalitarian communities to hierarchically complex societies is inherently linked with a rise in inequality, whether this relates to wealth, skills, or other attributes

(Bowles et al., 2010; Flannery, 2012; Johnson and Earle, 2000; Mattison et al., 2016). Sharing rules appears to have evolved jointly with this development. For example, food sharing is a common cooperative behaviour in hunter-gatherers. While the exact sharing behaviour seems to be influenced by ecological and economic factors, there is evidence that in relatively homogeneous communities, food is shared equally. As communities become more heterogeneous, however, sharing behaviour appears to become more responsive to differences in contributions (Bird et al., 2002). In this context, sharing might be better understood as a mutualistic relationship, like collaboration, rather than a consequence of pure altruism. As is argued in Newton (2017), such collaborations might be the driving factor of cooperative actions in human groups. Understanding the relation between inequality and cooperation thus seems an important research objective, to which this study aims to contribute.

Generally speaking, existing evidence relating to the effect of inequality on cooperative behaviour appears ambiguous. There is evidence from the experimental literature that people dislike inequality and prefer to equalise outcomes through re-distribution, which would also imply a preference for equal sharing (Alesina et al., 2002; Fehr and Schmidt, 2001). However, individuals in these settings are often ex-ante homogeneous and there is an ongoing debate in the literature on the interpretation of this evidence. Engelmann and Strobel (2004), for instance, show that fairness concerns might be in line with maximin preferences, while inequity aversion fails to accurately explain individual behaviour. Furthermore, the observed effects of inequality differ depending on the exact form of the social dilemma. In the context of the use of natural resources, the use of common resources becomes more efficient as inequality grows, while voluntary contributions to a common good decline (Baland and Platteau, 1999, 2018). In agreement with the latter, empirical studies show that contributions to local public goods are decreasing, as communities become more unequal (Alix-Garcia and Harris, 2014; Dayton-Johnson, 2000). In team contests, as inequality increases, rich players lose their incentives to contribute more than others which reduces team success in comparison to more homogeneous teams (Heap et al., 2015). Our study specifically examines the role of the distribution of payoffs to disentangle the effects of inequality as such, and the effects of egalitarian sharing. While inequality might also affect social preferences around reciprocity and egalitarianism (see for example Alesina et al. (2012); Alesina and Rodrik (1994); Attanasio et al. (2012); Bowles and Gintis (2011); Cox (2004)), we abstract from such considerations to focus on the link between inequality and unequal sharing.

Attempts to clarify the link between inequality, sharing, and cooperation in social dilemmas are not new. Van Dijk and Wilke (1995) present experimental evidence that, in a one-shot game, inequality in endowments leads to inequality in players' contributions, and similarly for inequality in shares of returns. Further experimental research has shown that unequal benefit distribution in combination with endowment inequality can have positive effects on contributions to a public good (Chan et al., 1999; McGinty and Milam, 2013). For instance, distributing profits according to individual effort in firms can increase performance, especially under peer pressure and mutual monitoring (DeMatteo et al., 1998; FitzRoy and Kraft, 1987; Kandel and Lazear, 1992). In addition, it is suggested that sharing according to individual effort should lead to efficient and fair market outcomes (Alesina et al., 2002). Following Olson's seminal analysis, it is often argued that under unequal sharing in a stage game, it is better if one group member is the one doing all the work (Olson, 1965). However, this is only true if players' contributions are perfectly substitutable (Hirshleifer, 1983). This is, for instance, not the case if players have different productivities. Ray et al. (2007) parametrise players' substitution and study its effect on the sharing rule in a joint project dilemma. They show that there exists a threshold for the elasticity of substitution of players under which the egalitarian sharing rule is superior. If the elasticity is above the threshold, then equal sharing can always be improved. In team competitions, where heterogeneous players form groups that compete for a prize, it was suggested that the sharing rule of the returns should be a mixture between the egalitarian division and division according to relative effort (Nitzan, 1991). If players are homogeneous, however, groups that share more equally tend to outplay their opponents (Nitzan and Ueda, 2011). Kugler et al. (2010) show experimentally that proportional sharing enhances team cooperation when competing with others. In addition, it was shown that the presence of a team leader that distributes the revenue from team production according to the individual contributions increases cooperative effort in comparison to egalitarian sharing (Karakostas et al., 2021; Van der Heijden et al., 2009).

Much of the previous work focuses on one-shot games, suitable to analyse behaviour in large communities. In smaller social groups, however, strategic incentives differ due to reciprocity and the higher chance of repeated interactions. Existing evidence confirms that in social dilemmas, individuals do take into account the repeated nature of interactions by reducing free-riding behaviour (Dal Bó, 2005). Our approach is thus particularly suited to analysing the effect of heterogeneity on cooperation in small groups. As an example, research teams are more likely to produce a high-impact work than individual researchers (Jones, 2021; Jones et al., 2008). Furthermore, there is evidence that heterogeneous teams tend to be more effective (Cheruvelil et al., 2014). How credit in research output is shared might thus affect the willingness of individual scientists to form teams and the success of such collaborations. To our knowledge, a systematic analysis of optimal sharing in heterogeneous groups is still lacking.

Perhaps most closely related to this objective, Kobayashi et al. (2016) study the optimal sharing rule in a two-player partnership game, in an extension of the model by Radner et al. (1986). In each period, both players decide to either work or shirk. Surplus is divided according to a fixed, previously agreed sharing rule. Players might differ in the effectiveness of their contribution to the project, and the cost of contributed time. While the core idea is closely related to our study, there are several critical differences. Our focus lies on studying the effect of endowment inequality on cooperation. Kobayashi et al. (2016) focus on the role of imperfect monitoring, while we assume perfect information in order to generalise results to n-players and more clearly highlight the impact of heterogeneity alone. The advantage of our approach lies in deriving the optimal sharing rule for a more general and widely adopted public good (or joint production) game. Our results coincide when interactions are among two equal players.

2 Model

Each period an agent i receives an endowment $e_i > 0$. The agent can invest a fraction $x_i \in [0,1]$ in a joint project and use the remaining fraction $(1 - x_i)$ for a private project which we also refer to as their outside option. Returns from both projects are realised and consumed at the end of the period. There are a total of n agents. The vector of all endowments is denoted by $\mathbf{e} = (e_1, ..., e_n)$. We take this endowment vector to be normalised such that $\sum_i e_i = 1$, which means it can be conveniently represented in a simplex Δ^n. Any return from the joint project is shared among the n agents according to a *sharing rule* $\mathbf{f} = (f_1, ..., f_n)$, where $f_i \in [0,1]$ is the fraction of the jointly produced good that agent i receives. For a sharing rule to be *feasible*, it has to fulfil the obvious restriction that $\sum_i f_i \leq 1$. Contributions to the joint project are not necessarily equally productive across agents. The contribution of an agent i achieves a rate of return r_i, called i's *productivity*. The vector $\mathbf{r} = (r_1, ..., r_n)$ denotes the productivities of all players. The effective rate of return of an agent i from an investment of an agent j is thus $f_i r_j$. For now, we assume that the productivity of the outside option is equal to 1 across all agents. While endowments and productivities are constant across time, contributions could potentially vary for each period $t \in \{1, 2, ...\}$. We denote these by the *contribution vector* $\mathbf{x}(t) = (x_1(t), ..., x_n(t))$, where $x_i(t) \in [0,1]$. After each period t, the game ends with probability $(1 - \delta)$, meaning the next period is reached with probability δ; equivalently referred to as the *continuation probability* or discount factor.
The utility an agent i receives in a period t given contributions $\mathbf{x}(t)$ of all players at t is:

$$u_i(\mathbf{x}(t)) = f_i \sum_{j=1}^{n} x_j(t) e_j r_j + \big(1 - x_i(t)\big) e_i$$

As only the ordinal properties of the utility function are relevant in this context, the previously mentioned normalisation of $\mathbf{e}$ remains without loss. The overall expected payoff of an agent i given some endowment $\mathbf{e}$ and set of contribution vectors $\{\mathbf{x}(t)\}_{t=1}^{\infty}$ can be written as:

$$\pi_i = (1 - \delta) \sum_t \delta^{t-1} u_i(\mathbf{x}(t)),$$

where $(1 - \delta)$ is a normalising factor. We can denote this as the game

$$\Gamma(\mathbf{e}, \mathbf{r}, \delta, \mathbf{f}) = \{N, X, \Pi\}$$

where $N = \{1, ..., n\}$ is the set of players, $X = X_1 \times ... \times X_n$ is the is the set of strategy profiles whereby X_i is the strategy set of player i, and $\Pi = \{\pi_1, ..., \pi_n\}$ denotes the players' payoff functions which depend on endowments $\mathbf{e}$, the productivities $\mathbf{r}$, continuation probability δ as well as the sharing rule $\mathbf{f}$.
We call a strategy profile an *equilibrium* of this game if it constitutes a subgame-perfect Nash equilibrium (SPE). As the action space of each player is convex and players are not risk-loving, we can restrict the

analysis without loss of generality to pure (non-stochastic) strategies.
The following definition makes precise what we mean by ‘social dilemma’. It is ‘social’ in the sense that the socially optimal outcome - the highest aggregate payoff - is achieved when all players invest their endowment in the joint production. This is ensured by part (i) of Definition 1. It constitutes a ‘dilemma’ as it is not individually rational for any player to invest by themselves; this is captured by (ii). This further implies that every player has an incentive to free ride on the contributions of the others.

Definition 1. *A game* $\Gamma(\mathbf{e}, \mathbf{r}, \delta, \mathbf{f})$ *is a social dilemma if for all* $i \in N$*: (i)* $r_i \geq 1$ *and (ii)* $f_i r_i < 1$.

As we are interested in settings where cooperation is socially optimal, we maintain the assumption that $r_i \geq 1, \ \forall i \in N$ throughout this analysis. This could, however, be relaxed for at least some i without affecting the results. With this assumption, aggregate payoffs are maximised when each player contributes their entire endowment to the joint project. We thus primarily focus on if and when this can be sustained in equilibrium. We call this ‘full cooperation’ of players. In other words, we characterise those games, for which $\mathbf{x}(t) = (1, ..., 1)$ is an equilibrium contribution vector for all t.

Definition 2. *We say the triple* $(\mathbf{e}, \mathbf{r}, \delta)$ ***allows for cooperation*** *if there exists a feasible sharing rule* $\mathbf{f}$ *such that full cooperation is an equilibrium in the game* $\Gamma(\mathbf{e}, \mathbf{r}, \delta, \mathbf{f})$.

A given triple $(\mathbf{e}, \mathbf{r}, \delta)$ allows for cooperation if there exists at least one way of sharing the surplus and punishing deviations such that no player prefers to free ride. Full cooperation can, of course, be trivially achieved in equilibrium if productivities are so high that players have an incentive to invest by themselves, independent of the others. For a sufficiently large $\mathbf{r}$, the returns from investing in the joint project can be shared according to some $\mathbf{f}$ such that $f_i r_i \geq 1$ for all players. It is easily verified that full cooperation is then a weakly dominant strategy at every t in the corresponding game Γ; which is not a social dilemma. To distinguish these trivial cases, we introduce the following definition:

Definition 3. *We say the triple* $(\mathbf{e}, \mathbf{r}, \delta)$ ***trivially allows for cooperation*** *if there exists a feasible sharing rule* $\mathbf{f}$ *in the game* $\Gamma(\mathbf{e}, \mathbf{r}, \delta, \mathbf{f},)$ *with continuation probability* δ *such that* $f_i r_i \geq 1, \ \forall i \in N$.

Even if $(\mathbf{e}, \mathbf{r}, \delta)$ trivially allows for cooperation, it is in principle possible to find an alternative sharing rule that creates a corresponding social dilemma. As will become clear, however, this would require that not all surplus from investing in the joint project is allocated to the players. Such an outcome is inefficient. The dilemma could be trivially resolved by (feasibly) increasing every player’s share. Interestingly, as the next result shows, this notion of triviality can be expressed independently of $\mathbf{e}$ and δ, only in terms of productivities $\mathbf{r}$. It thus suffices to check $\mathbf{r}$ in order to determine whether $(\mathbf{e}, \mathbf{r}, \delta)$ is trivial in the sense of Definition 3.

Lemma 1. *A triple* $(\mathbf{e}, \mathbf{r}, \delta)$ *trivially allows for cooperation if and only if* $\sum_i \frac{1}{r_i} \leq 1$.

We henceforth restrict the analysis to non-trivial cases, where cooperation is nevertheless socially optimal. This is formalised by the following assumption, which is maintained throughout.

Assumption 1. *Any productivity vector* $\mathbf{r}$ *is such that* $\sum_i \frac{1}{r_i} > 1$ *and* $\mathbf{r} \geq 1$*, with* $r_i > 1$ *for at least some* $i \in N$*.*

Under Assumption 1, if $(\mathbf{e}, \mathbf{r}, \delta)$ allows for cooperation, then it does so in a non-trivial way. Furthermore, aggregate payoffs from full cooperation exceed those of any equilibrium where not all player contribute their entire endowment. Cooperation is socially optimal.

3 Optimal sharing

3.1 Achieving full cooperation

Whether full cooperation can be achieved in equilibrium depends on how players react to a possible deviation by some player i from $x_i(t) = 1$. If this has no consequences on the actions of the others, given the social dilemma aspect of the game, a player would strictly prefer to lower their contributions and thus free ride. Such a deviation to some $x_i'(\tau) < 1$ at some time τ is not optimal for player i if the payoff on the equilibrium path exceeds that of a deviation. This requires:

$$\sum_t \delta^t u_i(\mathbf{x}(t)) \geq \sum_t \delta^t u_i(\mathbf{x}'(t)), \tag{1}$$

where $\mathbf{x}'(t)$ is the vector of contributions given the deviation of i at τ, including the response to this deviation at all $t > \tau$. To deter free riding, the actions players take after a deviation should reduce the payoff of the defector as much as possible. Clearly, the harshest such 'punishment' is achieved if $x_j'(t) = 0$ for all $j \neq i$ and $t > \tau$. Furthermore, this punishment is subgame-perfect in a social dilemma, as it is itself a Nash equilibrium of the stage game. In line with the literature, we call this trigger-strategy *Grim*. As this minimises the right-hand side of (1) for all periods after a deviation, it follows immediately that if full cooperation can be sustained for any strategy profile, it can be sustained under strategy Grim. It thus forms the baseline for the following analysis.

Using (1) and the definition of the strategy *Grim*, we can conclude that full cooperation is an equilibrium only if:

$$f_k \sum_{i \in N} e_i r_i \geq (1-\delta) f_k \sum_{j \neq k} e_j r_j + e_k, \quad \forall k \in N.$$

The left-hand side is the (equilibrium) payoff from full cooperation while the right-hand side is the payoff from the most profitable deviation ($x_i(t) = 0$). Equivalently, this can be written as:

$$\delta \sum_{j \neq k} e_j r_j \geq \big(\frac{1}{f_k} - r_k\big) e_k, \quad \forall k \in N \tag{2}$$

which can be re-arranged to derive a necessary condition for a sharing rule to sustain full cooperation under the strategy Grim:

$$f_k \geq \frac{e_k}{\delta \sum_{j \neq k} e_j r_j + e_k r_k}, \quad \forall k \in N. \tag{IC}$$

For a given set of parameters, IC characterises all potential ways of distributing the returns from the joint project such that no player has an incentive to deviate under the strategy Grim. It describes a set of sharing rules that can sustain full cooperation as Nash equilibrium. For *Grim* to be not just a Nash equilibrium but also subgame perfect, the off-equilibrium (punishment) path needs to be consistent with each player's incentives as well. In other words, not contributing to the joint project when no other player contributes needs to be a Nash equilibrium. We require:

$$f_k r_k < 1, \quad \forall k \in N. \tag{SD}$$

For any social dilemma, SD is satisfied by definition. We are thus interested in the subset of (feasible) sharing rules that constitute a social dilemma in the corresponding Γ. This set can be characterised as follows:

$$\mathcal{F}(\mathbf{e}, \mathbf{r}, \delta) = \{\mathbf{f} \mid \mathbf{f} \in \mathbb{R}^n \text{ is feasible and satisfies IC \& SD}\}. \tag{3}$$

Any element of $\mathcal{F}$ (if any) describes a possible way to share the returns from the joint project such that full cooperation is an equilibrium in the corresponding social dilemma Γ, given strategy Grim. Lemma 2 shows that this focus on social dilemmas is not a restriction. It implies that if full cooperation can be achieved in any equilibrium, there exists a feasible sharing rule $\mathbf{f}$ such that $\Gamma(\mathbf{e}, \mathbf{r}, \delta, \mathbf{f})$ is a social dilemma and full cooperation is an equilibrium for the strategy Grim.

Lemma 2. *If $(\mathbf{e}, \mathbf{r}, \delta)$ allows for cooperation, then $\mathcal{F}(\mathbf{e}, \mathbf{r}, \delta) \neq \emptyset$.*

Lemma 3 provides a convenient characterisation of those $(\mathbf{e}, \mathbf{r}, \delta)$ that allow for cooperation and thus have a corresponding $\mathcal{F}$ that is non-empty.

Lemma 3. *A triple $(\mathbf{e}, \mathbf{r}, \delta)$ allows for cooperation if and only if:*

$$\sum_{k \in N} \frac{e_k}{\delta \sum_{j \neq k} e_j r_j + e_k r_k} \leq 1. \tag{4}$$

If (4) is strict, then there is a continuum of sharing rules in $\mathcal{F}(\mathbf{e}, \mathbf{r}, \delta)$, meaning $|\mathcal{F}| = \aleph_1$. If (4) holds with equality, then $|\mathcal{F}| = 1$.

For any endowments $\mathbf{e}$, productivities $\mathbf{r}$, and continuation probability δ, the returns from the joint project can be shared such that full cooperation can be achieved in some equilibrium as long as (4) is satisfied. In combination with Lemma 2, this implies that if (4) holds, then full cooperation is an equilibrium under Grim in some social dilemma $\Gamma(\mathbf{e}, \mathbf{r}, \delta, \mathbf{f})$, where $\mathbf{f}$ is an element in the corresponding $\mathcal{F}$. For a generic set of parameters that allow for cooperation, however, there is more than one way of sharing to achieve cooperation. The following sections aim to further characterise $\mathcal{F}$ and some of the elements in it.

3.2 Possibility of cooperation

If productivities and the chance of future interactions are high, cooperation should be easy to sustain. In environments where the return from cooperation is slim and interactions are rare, constraints on equilibrium cooperation are harder to satisfy. There might be no way of sharing returns, such that individuals cooperate in equilibrium; $\mathcal{F}$ is empty.
Other than one might expect, however, Theorem 1 establishes that whether an environment permits cooperation is not entirely determined by $\mathbf{r}$ and δ. Even in those environments 'unfavourable' to cooperation (i.e., low $\mathbf{r}$ and/or δ), cooperation can be achieved at least for some endowment distributions. Interpreting this more loosely, if a jointly produced good can be flexibly shared among contributors, then we can find some contributions and sharing rule that makes cooperation the best course of action for everyone.

Theorem 1. *For any productivities $\mathbf{r}$ and continuation probability δ, there exists an endowment $\mathbf{e}$, such that $(\mathbf{e}, \mathbf{r}, \delta)$ allows for cooperation.*

While cooperation might not be achievable for all endowments, no matter how low δ and $\mathbf{r}$ are (given all assumptions are still satisfied), returns can be shared in a way that cooperation in equilibrium is possible for at least some $\mathbf{e}$, meaning $\mathcal{F}(\mathbf{e}, \mathbf{r}, \delta) \neq \emptyset$.
This does, however, rely on players being able to share flexibly. If we restrict attention to equal sharing (i.e., $f_k = f_{\text{eq}} \equiv 1/n$, $\forall k \in N$), it is easy to see that this does not hold. A particularly instructive case is when productivities are equalised across players (i.e., $r_k = r, \forall k \in N$). If for such an $\mathbf{r}$, sharing returns equally can make players cooperate for any endowment distribution, then it can do so for an equal endowment distribution (i.e., $e_k = e_{\text{eq}} \equiv 1/n, \forall k \in N$). In the words of Hauser et al. (2019), $\mathbf{e}_{\text{eq}}$ is the endowment distribution 'most conducive to cooperation'. However, there exist low enough $\mathbf{r}$ and δ such that $\mathbf{f}_{\text{eq}}$ fails to achieve full cooperation in any equilibrium. In fact, we can find $\mathbf{r}$ and δ such that $\mathcal{F}(\mathbf{e}_{\text{eq}}, \mathbf{r}, \delta) = \emptyset$. Such a triple $(\mathbf{e}_{\text{eq}}, \mathbf{r}, \delta)$ does not allow for cooperation. It then follows from the previous argument that $\mathbf{f}_{\text{eq}} \notin \mathcal{F}(\mathbf{e}, \mathbf{r}, \delta)$ for any $\mathbf{e}$. In other words, equal sharing is particularly effective at achieving cooperation for equal endowments and productivities. However, if it cannot achieve cooperation for equal endowments (and productivities), it also fails to entice players to cooperate for any other endowment distribution.
Interestingly though, as Proposition 1 shows, this is not at all representative of the set $\mathcal{F}$. In fact, we reach the opposite conclusion: if there is a way to share returns such that individuals are willing to cooperate at equal endowments, then this is also true for any other endowment.

Proposition 1. *Suppose that endowments $\mathbf{e}$ and productivities $\mathbf{r}$ are equal across players and $(\mathbf{e}, \mathbf{r}, \delta)$ allows for cooperation. Then for every endowment distribution $\hat{\mathbf{e}}$, every $\hat{\mathbf{r}} \geq \mathbf{r}$, and every $\hat{\delta} \geq \delta$, $(\hat{\mathbf{e}}, \hat{\mathbf{r}}, \hat{\delta})$ allows for cooperation.*

If we now choose the continuation probability and productivities such that cooperation becomes just feasible at equal endowments ($|\mathcal{F}(\mathbf{e}_{eq}, \mathbf{r}, \delta)| = 1$), then this is the only point in the endowment simplex where equal sharing can sustain cooperation. For any other $\mathbf{e}$, we have $\mathbf{f}_{\mathbf{eq}} \notin \mathcal{F}(\mathbf{e}, \mathbf{r}, \delta)$. However, with the same δ and $\mathbf{r}$, $\mathcal{F}(\mathbf{e}, \mathbf{r}, \delta) \neq \emptyset$ for any $\mathbf{e} \in \Delta^n$. As Figure 2 demonstrates, cooperation can be achieved at any endowment if players share appropriately and unequally.

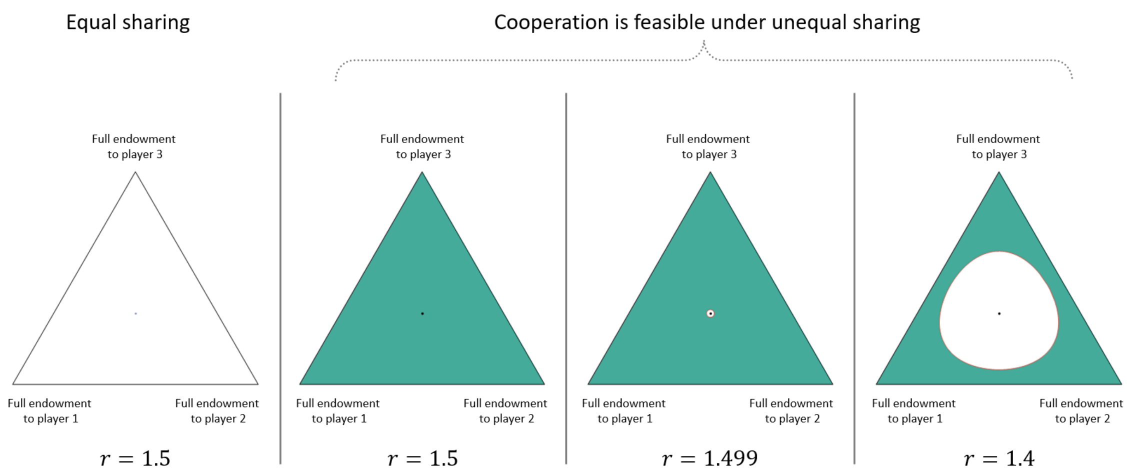

Figure 2: Possibility of full cooperation under different sharing rules for equal productivities. All plots are implemented for $\delta = 0.5$. Full cooperation is feasible under equal sharing only at $\mathbf{e} = (1/3, 1/3, 1/3)$, whereas unequal sharing allows for full cooperation $\forall \mathbf{e}$.

3.3 Endowment inequality and cooperation

The previous section highlighted that understanding the connection between endowments and cooperation is key in determining if and when cooperation can be achieved. Figures 2 and 3 give an indication that this link is not arbitrary but closely related to endowment inequality. This section explores this link systematically. We first focus on comparing endowments regarding their 'cooperativeness' in general before analysing the effects of inequality in particular.

Definition 4. *Suppose for some endowments* $\mathbf{e}$ *and* $\hat{\mathbf{e}}$*, and productivities* $\mathbf{r}$*, there exists a discount factor* δ*, such that* $(\hat{\mathbf{e}}, \mathbf{r}, \delta)$ *allows for cooperation but* $(\mathbf{e}, \mathbf{r}, \delta)$ *does not. Then we say for productivities* $\mathbf{r}$*,* ***cooperation is easier to sustain*** *with the endowment vector* $\hat{\mathbf{e}}$ *than* $\mathbf{e}$*.*

Definition 4 establishes a particular ordering regarding the cooperativeness of endowments for a given set of parameters. If for one endowment players can find a way of sharing to sustain cooperation ($\mathcal{F}(\hat{\mathbf{e}}, \mathbf{r}, \delta) \neq \emptyset$), while for the other they can't ($\mathcal{F}(\mathbf{e}, \mathbf{r}, \delta) = \emptyset$), then the former endowment is in some sense more cooperative at these parameter values. But even though the comparison is defined at a specific $\mathbf{r}$ and δ, it is more general than it might appear due to how $\mathcal{F}$ depends on $\mathbf{r}$ and δ.
We can see that the constraint IC is relaxed as $\mathbf{r}$ and δ increase, as is (4). It follows then from Lemma 3 that if $(\hat{\mathbf{e}}, \mathbf{r}, \delta)$ allows for cooperation, then this is also true for any higher productivities and continuation

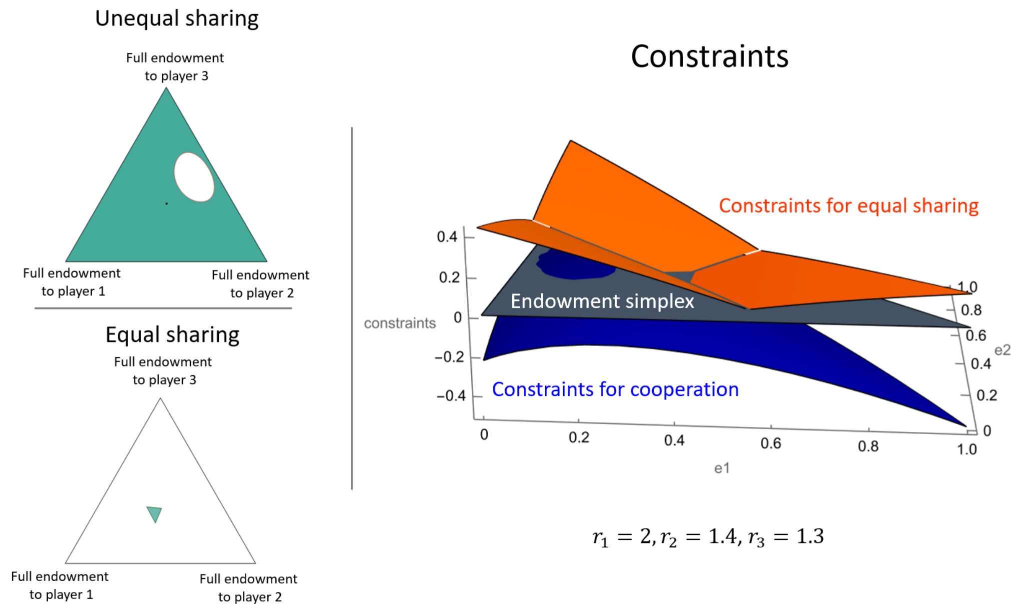

Figure 3: Possibility of full cooperation under different sharing rules for unequal productivities. All plots are implemented for $\delta = 0.5$.

probability. So if at some $\mathbf{r}$, cooperation is easier to sustain with $\hat{\mathbf{e}}$ than $\mathbf{e}$, the minimum continuation probability $\delta_{\min}$ required to achieve cooperation with these productivities is necessarily lower for $\hat{\mathbf{e}}$ than $\mathbf{e}$. Furthermore, for any continuation probability greater than this minimum, cooperation can be achieved with $\hat{\mathbf{e}}$. In this sense, $\hat{\mathbf{e}}$ makes cooperation easier. We can then ask if given some productivity vector, is there an endowment vector with which cooperation is easier to sustain than with any other endowment. In other words, is there a *maximally cooperative* endowment in the simplex? Under optimal sharing, this turns out to be not the case.

Proposition 2. *No maximally cooperative endowment exists. That is, for any* $\mathbf{e}$ *and* $\mathbf{r}$*, there exists an endowment* $\hat{\mathbf{e}}$ *with which cooperation is easier to sustain for* $\mathbf{r}$*. Furthermore, there exists* $\hat{\mathbf{r}} < \mathbf{r}$ *such that this is still true.*

As can be easily seen from IC, the minimum share a player need to receive from the joint project increases with the player's endowment. A player who invests more, all else equal, has a higher minimum compensation threshold, below which cooperation cannot be achieved. Whether full cooperation is an equilibrium outcome thus depends on how endowments are distributed. Interestingly though, Proposition 2 shows that no matter the individual productivities of agents, there is no optimal way to distribute endowments in order to maximise cooperation for the widest possible range of parameters. One direct implication is that if we could choose the endowment distribution from some compact subset of the simplex, then the choice that maximises the cooperativeness of the endowment allocation would always lie

on the boundary of this set. In other words, possible re-distributions should be fully realised (in some direction) when maximising cooperativeness.
Generally speaking, this arises because more unequal endowments make cooperation easier and there are no 'most unequal' endowments that are still a social dilemma, unless the set of endowments is closed. Theorem 2 generalises the insight from Proposition 5, that endowment inequality can promote cooperation. It shows that for any productivity vector and no matter the endowments, there exists at least some increase in inequality of endowments that makes defection less likely. As will be shown, this does not arise from any effect on the average productivity, but it is the inequality itself that is driving this effect. To formalise this discussion, we introduce the following definition that makes precise how endowments can be compared in terms of their (in-)equality.

Definition 5. *An endowment* $\hat{\mathbf{e}}$ *is* ***more unequal*** *than* $\mathbf{e}$ *if the distribution of* $\hat{\mathbf{e}}$ *is a mean-preserving spread of the distribution of* $\mathbf{e}$*.*

A formal definition of a mean-preserving spread is provided in Mas-Colell et al. (1995), p.197. Intuitively, an endowment $\hat{\mathbf{e}}$ is more unequal than $\mathbf{e}$, if $\hat{\mathbf{e}}$ can be obtained from $\mathbf{e}$ by transfer of endowment from agents with lower endowment to agents with higher endowment. Given that only endowment shares rather than absolute values are relevant for equilibrium behaviour, various alternative definitions could be implemented without affecting the results.

Theorem 2. *For any productivities* $\mathbf{r}$ *and endowment* $\mathbf{e}$*, there exists a more unequal endowment* $\hat{\mathbf{e}}$*, such that for* $\mathbf{r}$*, cooperation is easier to sustain with the endowment vector* $\hat{\mathbf{e}}$ *than* $\mathbf{e}$*.*

Proposition 3 furthermore shows that this result does not hinge on the productivity of any particular player. It is inequality itself that makes cooperation easier, no matter which player holds the majority of the endowment.

Proposition 3. *For any productivity vector and continuation probability, full cooperation can be sustained with any sufficiently unequal endowment distribution. That is, for any* $\mathbf{r}$ *and* δ*, there exists* $\epsilon \in (0,1)$ *such that for any endowment* $\mathbf{e}$ *with* $\max\{e_i\}_{i\in N} > 1-\epsilon$*,* $(\mathbf{e},\mathbf{r},\delta)$ *allows for cooperation.*

3.4 Towards an optimal sharing rule

As Lemma 3 established, for a generic $(\mathbf{e},\mathbf{r},\delta)$ that allows for cooperation, there exist a continuum of sharing rules in $\mathcal{F}$. In these cases, (4) holds with strict inequality. The remaining returns from the joint project can be freely allocated among agents without violating any constraints. Only as δ and/or $\mathbf{r}$ decrease, does (4) provide a tight characterisation.
For any given endowment and productivity vector, there exists a lower limit on δ below which cooperation cannot be implemented in equilibrium. The incentive to defect becomes too high as the probability that the interaction is repeated vanishes. At the δ_{min} where cooperation can just be achieved, the corresponding $\mathcal{F}$ contains exactly one element. This sharing rule will receive particular attention in the subsequent analysis.

For a given $\mathbf{e}$ and $\mathbf{r}$, we denote the lowest δ such that $(\mathbf{e}, \mathbf{r}, \delta)$ allows for cooperation by $\delta_{\min}$. Furthermore, as follows from Lemma 3, $|\mathcal{F}(\mathbf{e}, \mathbf{r}, \delta_{\min})| = 1$. We denote this element of $\mathcal{F}$ by $\mathbf{f}^*$. It follows from the definition of $\mathcal{F}$ that if there is only one element, then IC must hold with equality for all players. This yields an expression for $\mathbf{f}^*$:

$$f_k{}^* \equiv \frac{e_k}{\delta_{\min} \sum_{j \neq k} e_j r_j + e_k r_k}, \ \forall k \in N. \tag{5}$$

As Lemma 4 shows, this sharing rule satisfies several desirable properties. First of all, it is efficient in the sense that it allocates all returns ($\sum_{k \in N} f_k^* = 1$). No surplus is wasted. Furthermore, for a given $\mathbf{e}$ and $\mathbf{r}$, the corresponding $\mathbf{f}^*$ ensures that $\Gamma(\mathbf{e}, \mathbf{r}, \delta, \mathbf{f}^*)$ is a social dilemma. And finally, if $(\mathbf{e}, \mathbf{r}, \delta)$ allows for cooperation (i.e., $\delta \geq \delta_{\min}$), then $\mathbf{f}^* \in \mathcal{F}(\mathbf{e}, \mathbf{r}, \delta)$. The sharing rule $\mathbf{f}^*$ can not only ensure cooperation in the corresponding social dilemma Γ for $\delta_{\min}$, but also for any larger δ.

Lemma 4. *If $(\mathbf{e}, \mathbf{r}, \delta)$ allows for cooperation, then there is at least one sharing rule $\mathbf{f}$ that sustains full cooperation in the social dilemma $\Gamma(\mathbf{e}, \mathbf{f}, \mathbf{r})$* ***and*** *$\sum_i f_i = 1$. $\mathbf{f}^*$ is such a rule.*

As is clear from (5) and the definition of $\delta_{\min}$, both are jointly determined.While they can be characterised from the set of necessary and sufficient conditions to maintain cooperation in equilibrium, this potentially poses problems for the computation of both when there is a large number of players. Proposition 5 provides a computationally convenient method to derive $\mathbf{f}^*$ and $\delta_{\min}$ for arbitrary n. As Figure 4 shows, qualitatively, $\delta_{\min}$ behaves similarly to the cooperation constraint itself: the more unequal the endowment distribution is, the lower the requirement on players' patience.

Proposition 4. *The optimal sharing rule $\hat{\mathbf{f}}$ is the eigenvector that corresponds to the largest eigenvalue of the matrix Φ such that*

$$\phi_{ij} = \begin{cases} \dfrac{-e_i(r_i - 1)}{\sum_{k \neq i} e_k r_k}, & i = j \\ \dfrac{e_i}{\sum_{k \neq i} e_k r_k}, & i \neq j \end{cases}$$

Moreover, the largest eigenvalue is the minimal continuation probability ($\delta_{\min}$) for this sharing rule to be feasible.

To gain some intuition for this result, note that equation (5) can be rewritten as

$$\underbrace{\delta_{\min}\Big(f_i^* \sum_{j \neq i} e_j r_j\Big)}_{\text{cost of punishment for defection}} \quad - \quad \overbrace{\Big(e_i - f_i^* e_i r_i\Big)}^{\text{private benefit from defection}} \quad = 0.$$

At the limit where cooperation can just be sustained, the private benefit from defection is exactly compensated by the cost of punishment. A higher individual share reduces the savings from not contributing to the joint project and increases the loss from forgone future contributions. More specifically, an agent

i who fails to contribute obtains a return e_i but forgoes their private return from the contribution to the joint project $f_i e_i r_i$. As f_i increases, this difference diminishes. Furthermore, this player also forgoes any future shares in the joint returns as the deviation triggers the punishment, reducing all future contribution to 0. A higher continuation probability puts further weight on the cost of this punishment as it makes it more likely that these periods are actually reached. The optimal sharing rule f^* together with the minimum continuation probability $\delta_{\min}$ balance these forces jointly for all players. This gives rise to a system of n equations, which can be written as:

$$\Phi \mathbf{f}^* = \delta_{\min} \mathbf{f}^*, \tag{6}$$

with Φ as defined in Proposition 4. As is clear from (6), $\mathbf{f}^*$ is an eigenvector of Φ, with $\delta_{\min}$ the corresponding eigenvalue. This provides a convenient method for jointly determining $\mathbf{f}^*$ and $\delta_{\min}$. The subsequent sections are devoted to the derivation and analysis of this sharing rule $\mathbf{f}^*$.

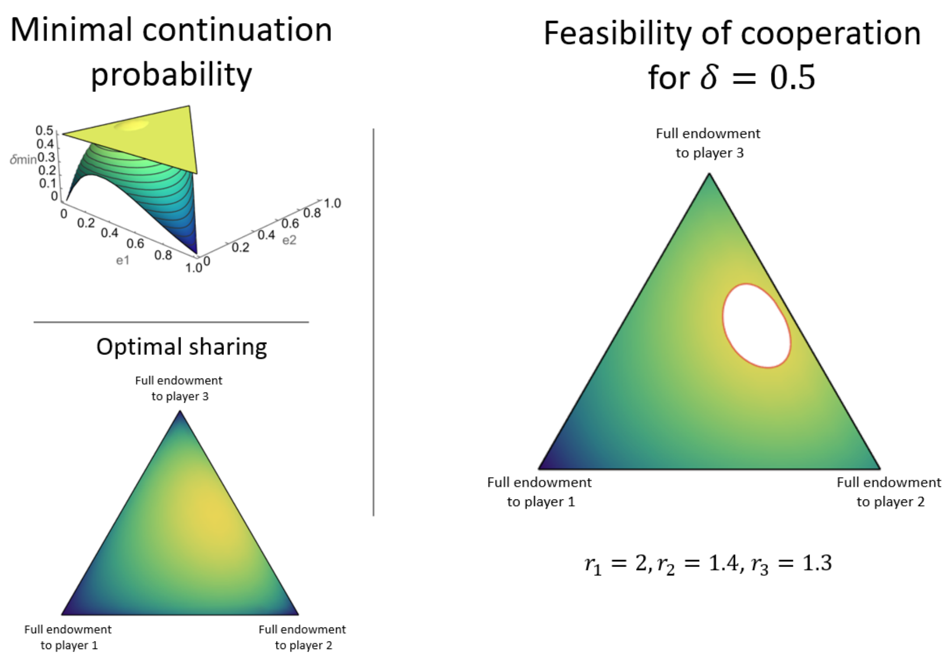

Figure 4: Optimal sharing rule and the minimum continuation probability as a function of players' endowments. On the right, we plot feasibility area of full cooperation under the optimal sharing rule for players' with continuation probability $\delta = 0.5$. There exists a set of endowments for which $\delta_{\min} > 0.5$ and cooperation cannot be sustained.

3.5 Comparison to other sharing rules

As a first step in analysing the properties of the optimal sharing rule, we contrast it against the egalitarian sharing rule $\mathbf{f}_{\text{eq}}$ that awards each player the same share $1/n$ independent of productivities and endowments. The corresponding $\Gamma(\mathbf{e}, \mathbf{r}, \delta, \mathbf{f}_{\text{eq}})$ is then a classic public goods game where returns are equal for all agents. The following Proposition 5 establishes that $\mathbf{f}^*$ does generically strictly better than equal sharing in maintaining cooperation: fixing some generic endowment and productivities, the corresponding $\mathbf{f}^*$ can maintain cooperation for (strictly) lower δ than equal sharing. This also holds for lower productivities, even though when reducing some r_j, a given fixed $\mathbf{f}^*$ is no longer optimal. Nevertheless, it can sustain full cooperation for some (lower) productivities where full cooperation under equal sharing fails.

Proposition 5. *If for some endowment distribution* $\mathbf{e}$*, productivities* $\mathbf{r}$*, and* $\delta > 0$*, full cooperation can be sustained as equilibrium under equal sharing, then it can be implemented with the sharing rule* $\mathbf{f}^*$*. For generic such* $\mathbf{e}$*,* $\mathbf{r}$*, and* δ*, there exists* $\tilde{\mathbf{r}} < \mathbf{r}$ *and* $\tilde{\delta} < \delta$ *such that full cooperation can be sustained under* $\mathbf{f}^*$ *but not under equal sharing.*

We can conclude that modelling a joint production task as a public goods game and thus imposing an equal sharing rule comes with a loss of generality. Cooperation is generically harder to maintain. The two approaches only coincide for a very specific cases, as is formally characterised by Proposition 6.

Proposition 6. *The optimal sharing rule is an equal sharing rule* ($\mathbf{f}^* = \mathbf{f}_{eq}$) *if and only if:*

$$\frac{\hat{e}_k}{\hat{e}_i} = \frac{n - (1 - \delta_{\min})r_i}{n - (1 - \delta_{\min})r_k}, \ \forall i, k. \tag{7}$$

An endowment distribution $\hat{\mathbf{e}}$ *satisfying* (7) *is non-generic.*

To compensate for the shortcoming of equal sharing with heterogeneous players, one might consider splitting the returns proportional to players' contributions. This has, for instance, been axiomatically proposed as a desirable sharing rule for linear production technologies in the context of cooperative game theory (Moulin, 2002). With linear technologies, as are considered here, the effect of any contribution on output is independent of other players' contributions. Proportional sharing thus weighs contributions equally. In our setup, this allows for two interpretations: sharing according to the actual contributions $x_k e_k$, or the effective contributions weighted by players' productivities, $x_k e_k r_k$. The corresponding sharing rules for full cooperation ($x_k = 1$) are thus defined as:

$$g_k^* = e_k \quad \text{and} \quad h_k^* = \frac{e_k r_k}{\sum e_j r_j}.$$

It can be easily verified that, for certain $\mathbf{e}$, $\mathbf{r}$, and a sufficiently high δ, both rules satisfy condition (IC) and can thus sustain full cooperation. While equal sharing fails to achieve full cooperation in equilibrium for high inequality, proportional sharing can, at least for some δ, overcome the incentive problem of richer players. However, it can also lead to the situation when Γ does not constitute a social dilemma anymore as full contribution becomes a dominant strategy for rich players even if poor players fail to contribute. The aim here is to better understand how the effects of inequality differ across these rules and particularly how they compare to the (strategically) optimal sharing rule derived in (5).

Proposition 7. *The proportional sharing rule g_k^* is equal to ${f_k}^*$ if and only if*

$$\frac{\hat{e}_k}{\hat{e}_i} = \frac{r_i}{r_k}, \ \forall i, k. \tag{8}$$

An endowment distribution $\hat{\mathbf{e}}$ *satisfying* (8) *is non-generic.*

Proposition 8. *The proportional sharing rule h_k^* is equal to ${f_k}^*$ if and only if*

$$\frac{\hat{e}_k r_k}{\hat{e}_i r_i} = \frac{(1-\delta_{\min} r_k) r_i}{(1-\delta_{\min} r_i) r_k}, \ \forall i, k. \tag{9}$$

An endowment distribution $\hat{\mathbf{e}}$ *satisfying* (9) *is non-generic.*

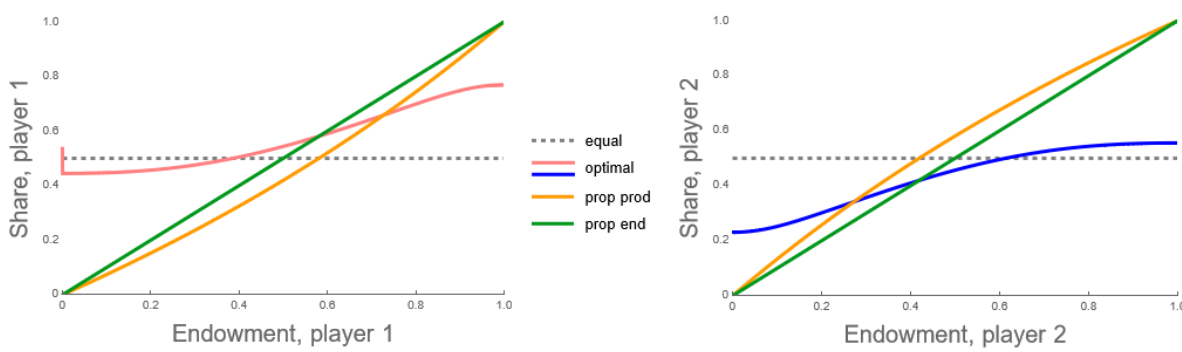

Figure 5: Proportional, proportional according to the productivities and optimal sharing rules for a 2-player game with $r_1 = 1.3$, $r_2 = 1.8$.

For example of a two-player game, the shares players receive under rules h_k^* and ${f_k}^*$ are equal if and only if

$$\frac{e_1}{e_2} = \frac{r_2 - 1}{r_1 - 1}.$$

For the set of productivities from Figure 5 such that $r_1 = 1.3$ and $r_2 = 1.8$ these sharing rules coincide at the endowment distribution $e_1 = 8/11$ and $e_2 = 3/11$. As can be seen in Figure 5 (left), the share player 1 receives as they get richer after $e_2 = 8/11$ does not increase. For the optimal rule, as players' endowments become more unequal, the marginal return from investing in a joint project, defined as $\hat{f}_i r_i$, approaches 1. However, the share of the richest player will still be less than 1. Even though the endowment distribution is such that $\frac{e_i}{e_j} \to \infty$, the ratio of shares remains finite. This implies that the poor player receives a disproportionately large share of the produced good.

In case of a three-player game, the exact outcome of all four sharing rules depend on the productivities of players. For equal sharing, inequality in endowments makes cooperation harder to sustain, and, when productivities are unequal, the largest endowment share should be given to the most productive player. For equal productivities, both proportional sharing rules exhibit identical behaviour as the equal sharing. As productivities of players become unequal, behaviour of these two rules departs from each other: sharing proportional to endowments guarantees a larger share to a more productive player, while sharing proportional to effective contributions guarantees a larger share to the least productive player. Generally, inequality in productivities makes cooperation harder to sustain for the sharing proportional to effective contributions. We demonstrate these observations in Figures 6 and 7. While it might appear that the proportional sharing according to players' endowments behaves similarly to the optimal sharing, one can find examples where both proportional rules fail at sustaining cooperation for any endowment distribution, yet equal and optimal sharing rule can sustain cooperation for at least some endowment distributions (e.g. for parameters $r_1 = 1.1,\ r_2 = 1.5,\ r_3 = 2.9,\ \delta = 0.3$).

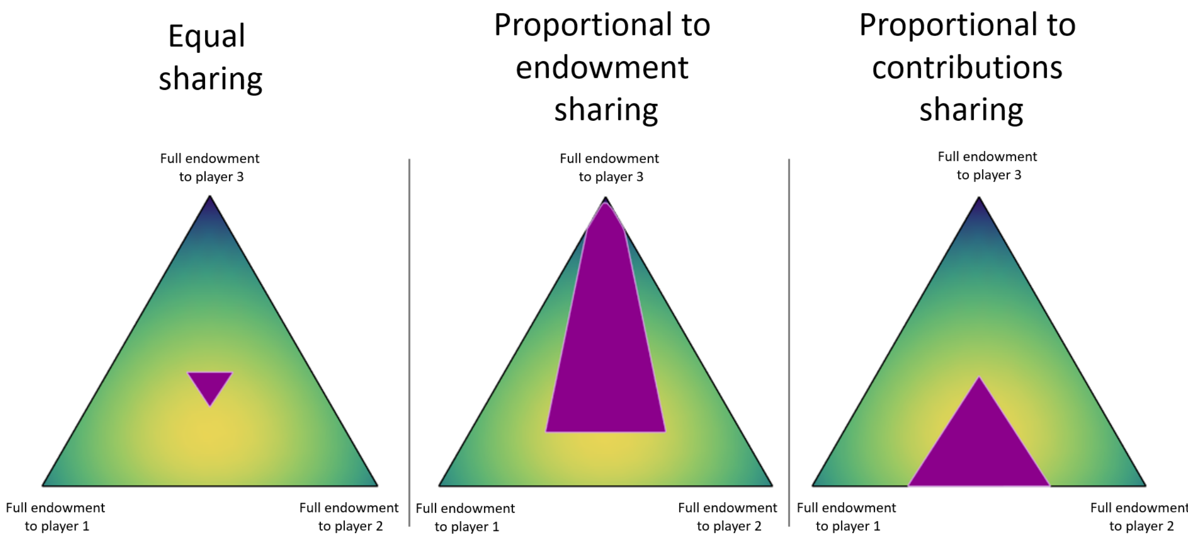

Figure 6: Possibility of cooperation (in purple) for $r_1 = r_2 = 1.4,\ r_3 = 1.7$ and $\delta = 0.6$. For this set of parameters, the optimal sharing allows for cooperation for any endowment distribution and is presented as a colormap filling of the endowment simplex: the darker the colour, the easier it is for the optimal sharing to sustain cooperation.

3.6 Heterogeneity of outside options

In the model presented so far, we accounted for possible heterogeneity of players in their endowments as well as their productivity regarding contributions to the joint project. The marginal return from their outside option - for example returns from private consumption - was normalised across players. However, one can easily imagine that players also differ in terms of the attractiveness of the outside option they have access to. As a specific example, players that are more productive in the joint production might also be more productive in other private tasks. Productivities would then be positively correlated across public and private production. Or alternatively, what might make players specifically suited for a particular task, might make them less productive in others. In this case, the correlation between the productivity in a joint task, r, and in a private project, q, would be negative.

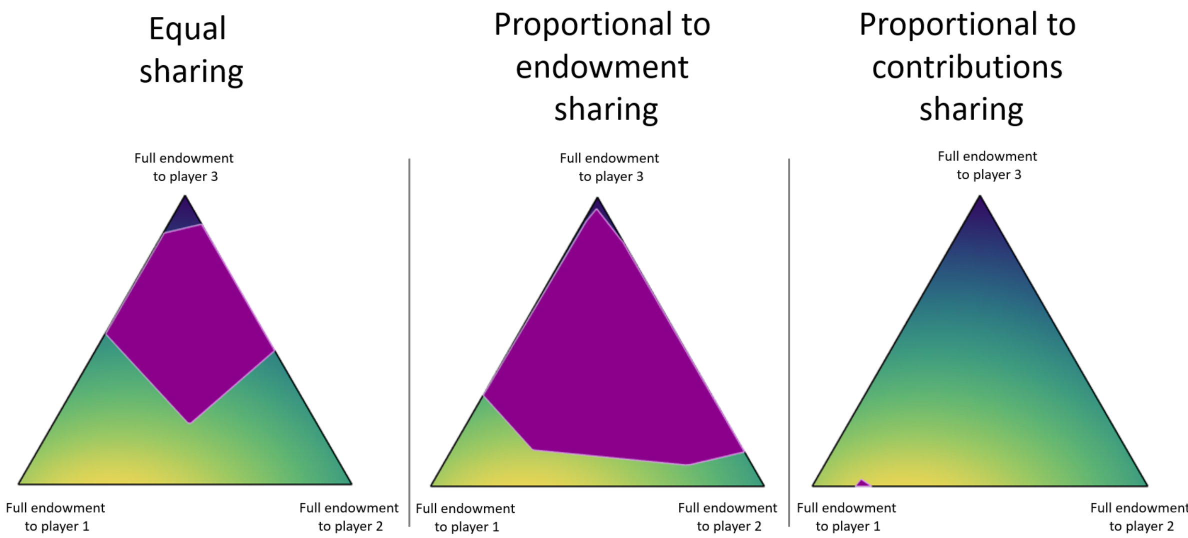

Figure 7: Feasibility of cooperation (in purple) for $r_1 = 1.1, \ r_2 = 1.5, \ r_3 = 2.9, \ \delta = 0.6$. For this set of parameters, the optimal sharing allows for cooperation for any endowment distribution and is presented as a colormap filling of the endowment simplex: the darker the colour, the easier it is for the optimal sharing to sustain cooperation.

In order to accommodate this possible variation into our model, we now allow returns from the private activity to vary across players. Denote by the vector $\mathbf{q}$ the return of the outside option of each player. For example, a player k with outside productivity q_k receives utility $q_k e_k(1 - x_k)$ from investing an amount $e_k - e_k x_k$ in their private project. In the previous analysis, q_k was simply assumed to be 1.

This additional heterogeneity naturally affects the minimum shares players have to be guaranteed in the joint project for full cooperation to be sustainable in equilibrium. A player with high q needs to be guaranteed a larger share, all else equal. More precisely, the minimum share of a player k for this player to cooperate in equilibrium takes the form:

$$f_k(q_k) \geq \frac{e_k q_k}{\delta \sum_{j \neq k} e_j r_j + e_k r_k} \tag{10}$$

That is, players' shares in the joint project have to cover potential benefits they could get by investing in other projects. However, given the multiplicative form of the rule (10), our previous results still hold (concavity in $\mathbf{e}$), i.e. our results are robust with respect to such a model modification.

When taking into account outside options, one can determine the sharing rule most conducive to cooperation in a similar manner as before by constructing matrix $\Phi(\mathbf{q})$ such that

$$\phi_{ij}(\mathbf{q}) = \begin{cases} \dfrac{-e_i(r_i - q_i)}{\sum_{k \neq i} e_k r_k}, & i = j \\ \dfrac{e_i q_i}{\sum_{k \neq i} e_k r_k}, & i \neq j. \end{cases}$$

The corresponding eigenvalue $\delta_{\min}(\mathbf{q})$ with $q_i \geq 1$ is greater than $\delta_{\min}$. Necessarily, cooperation becomes harder to sustain as outside options become more valuable.

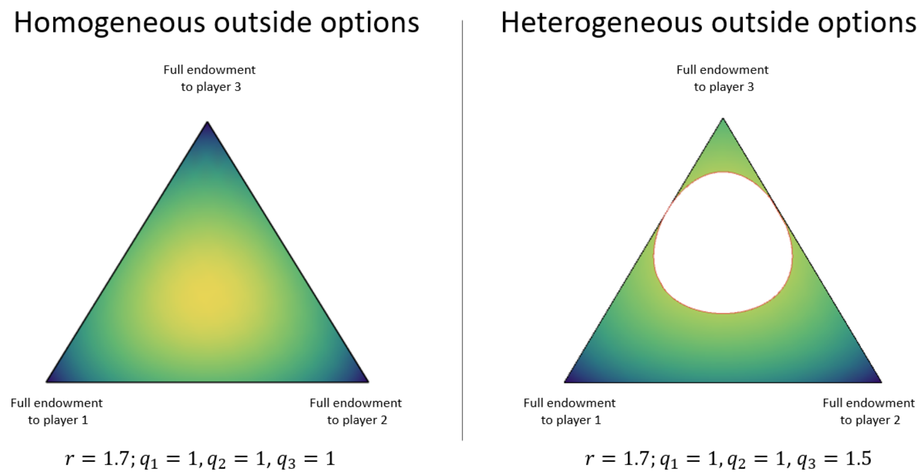

Figure 8: Feasibility of full cooperation under sharing rules with homogeneous and heterogeneous outside options. All plots are implemented for $\delta = 0.5$.

4 Evolutionary dynamics under the optimal sharing rule

In order to verify predictions of the model, we run evolutionary simulations using introspection dynamics, introduced in Hauser et al. (2019). As is usually the case, we study the evolution of strategies to determine the evolution of behaviour of players. In other words, we examine which strategies are more likely to be adopted by the population of players and observe the resulting choices over time. In order to model the reciprocity implied by our setup and yet limit complexity, we restrict our analysis to memory-one strategies, which take into account the outcome of the previous round but not earlier rounds. Memory-one strategies are simple enough to be analysed analytically and via numerical simulations (Baek et al., 2016; Hilbe et al., 2017), yet complex enough to capture the most frequently observed behavioural strategies like Defectors, Cooperators, Grim players or Conditional cooperators (strategy Tit-for-Tat) (Dal Bó and Fréchette, 2018, 2019; Fischbacher et al., 2001). These strategies can be represented by a vector $\mathbf{p}^i = (p_{0,\hat{x}^i}, p_{\mathbf{x},\hat{x}^i})$. Here, $p_{0,\hat{x}^i}$ is the probability that player i contributes $\hat{x}^i$ in round 0. The entries $p_{\mathbf{x},\hat{x}^i}$ denote the conditional probability that player i contributes $\hat{x}^i$ in any subsequent round given that the contribution in the previous round was $\mathbf{x}$. To make sure the strategies are stochastic, we require $\sum_{\hat{x}^i} p_{0,\hat{x}^i} = 1$ and $\sum_{\hat{x}^i} p_{\mathbf{x},\hat{x}^i} = 1$. In the simplest case, the contributions $\hat{x}^i$ can be either 0 or 1, which would correspond to the action Defect or Cooperate in a Prisoner's Dilemma setup. Such strategies are called deterministic. However, in our simulations, we consider strategies such that $\hat{x}^i$ can take any value in the interval $[0, 1]$.

Memory-one strategies allow us to calculate payoffs by constructing a Markov chain, M. Each state of M corresponds to possible outcomes of the previous round. For example, if we consider a game between two players with deterministic strategies, the following outcomes are possible: both players cooperate, player 1 defects and player 2 cooperates, player 1 cooperates and player 2 defects, or both players defect. We construct a transition Markov matrix $M = (m_{\mathbf{x},\mathbf{x}'})$, where $m_{\mathbf{x},\mathbf{x}'}$ is the set of states defined as

$$m_{\mathbf{x},\mathbf{x}'} = \prod_i p^i_{\mathbf{x},\hat{x}^i}.$$

Here, $\mathbf{x}$ is the contribution vector in the current round and $\mathbf{x}'$ is the contribution vector in the next round. Let $v^0_{\hat{x}} = \prod_i p^i_{0,\hat{x}}$ be the probability that players contribute $\hat{x}$ in the very first round. Then, the invariant distribution of the Markov chain M can be calculated as

$$\mathbf{v} = (1-\delta)\mathbf{v}^0 \cdot (I - \delta M)^{-1},$$

where $\mathbf{v}^0 = (v^0_{\hat{x}})$ and I is the identity matrix. This invariant distribution denotes the probabilities to observe contributions $\hat{x}$ over the infinite number of interactions given the continuation probability δ. Given these probabilities, we can calculate the average payoffs of players as

$$\pi_i = \sum_{\mathbf{x}} v_{\mathbf{x}} \cdot u_i(\mathbf{x}).$$

As $\delta \to 1$, the vector $\mathbf{v}$ approaches a left eigenvector of M. In order for this vector to be unique and for the matrix M to be ergodic, we allow for non-zero probabilities of errors such that players every now and then execute a different strategy. Specifically, we consider $\mathbf{p}_\epsilon = (1-\epsilon)\cdot\mathbf{p} + \epsilon\cdot(\mathbf{1}-\mathbf{p})$.

In the introspection dynamics (Hauser et al., 2019), at every time step a player is chosen at random to update their strategy. Let the chosen player to use strategy $\mathbf{p}^i$ before updating. The chosen player adopts a new strategy $\tilde{\mathbf{p}}^i$ with the probability

$$\rho_s = \frac{1}{1+\exp[-s(\pi_i - \tilde{\pi}_i)]},$$

where s is the parameter describing the strength of selection and π_i and $\tilde{\pi}_i$ are the corresponding payoffs for strategies $\mathbf{p}^i$ and $\tilde{\mathbf{p}}^i$. In the simulations, at every time step, one player is chosen randomly to compare their current strategy to a new randomly generated stochastic memory-one strategy. With probability ρ_s the chosen player switches to the new strategy. Afterwards, the next time step begins.

Results of computer simulations for three players can be found in Figure 9. Generally, predictions of the simulations parallel predictions of our model, that is, inequality makes cooperation easier to sustain. However, even though for this set of parameters the model predicts that full cooperation is feasible for any endowment distribution, in the stochastic simulations players were less likely to cooperate for sufficiantly high equality, that is, in the centre of the simplex. This can be due to the inability to fixate on an equilibrium, where effort is required from all three players due to a frequent updating events.

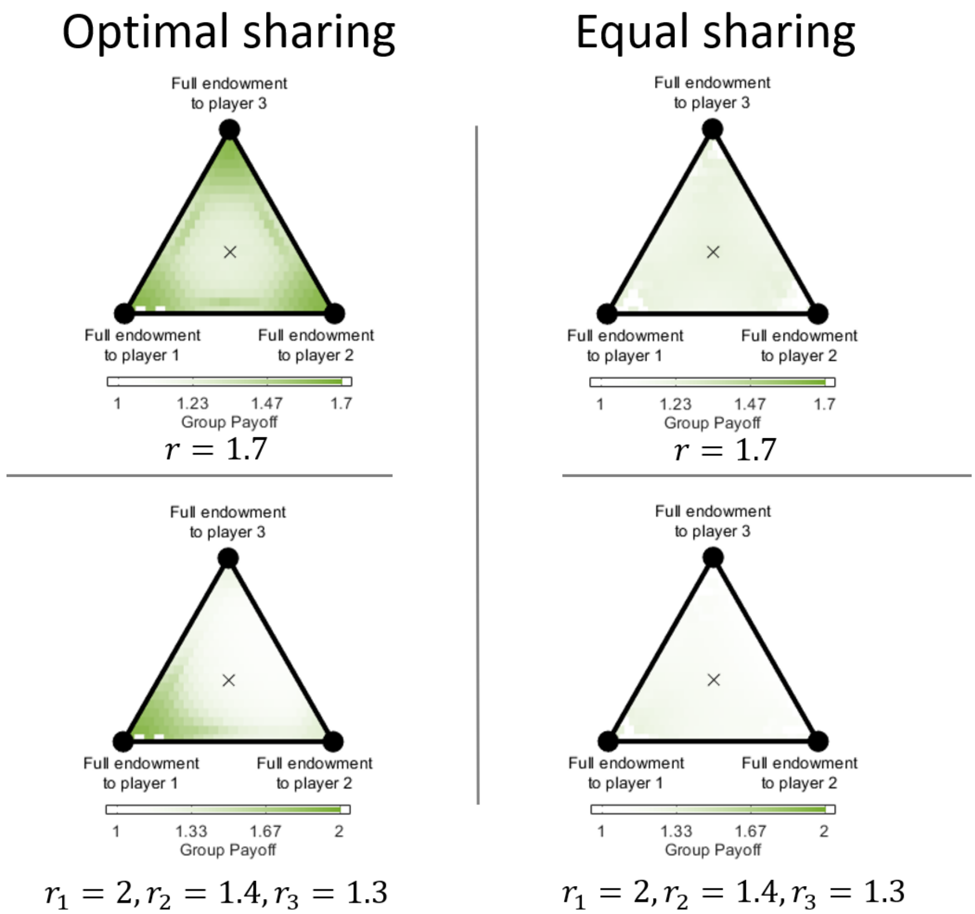

Figure 9: Evolutionary simulations for selection strength $s = 1,000$, number of generations 10^6 and $\delta = 1$.

5 Discussion and conclusions

We systematically address the sustainability of (full) cooperation in infinitely repeated interactions among heterogeneous agents. Broadly speaking, inequality of players has previously been seen as detrimental for cooperation in social dilemmas. However, we argue that this is specific to an egalitarian sharing of the returns. We show that in social dilemmas, sharing returns according to individual characteristics avoids the destabilising effect of inequality that is observed under egalitarian sharing. If individual shares appropriately account for endowments, productivities, and strategic incentives of contributors, heterogeneity of players can lead to more efficient social outcomes. While there might be larger societal benefits to reducing inequality, our results suggest inequality and heterogeneity might not have as negative an influence as might be expected. In the context of research collaborations, for instance, this suggests that

heterogeneity of contributors might benefit the joint project if credit can be attributed accordingly.
While we find there is generically no unique sharing rule to facilitates full cooperation in equilibrium, we characterise a (unique) sharing rule as a function of endowments and productivities that performs weakly better than all other such rules. We analyse the properties of this rule, some of which might be unexpected. Players with the largest endowments, for instance, may be compensated proportionally less than much lower contributors. Furthermore, sufficient inequality in endowments allows for full cooperation in settings that would otherwise only result in defection. As an implication of this result, it is generally beneficial for cooperation, and thus welfare, if one individual takes a leading role in a project, even if individuals are otherwise identical. Furthermore, heterogeneity in productivity does not require an 'aligned' inequality in endowments for this benefit to take effect. For example, suppose collaborators in a team can pledge the available time that they, in principle, invest in a project. Then maximising the chance of everyone successfully following through does not necessarily require the member with the highest productivity to invest the most time. It is, however, beneficial if time allocations differ rather than everybody contributing equally - provided that returns are shared in accordance with everyone's contributions and strategic incentives. Hence, with flexible credit sharing, collaborations might succeed because of and not in spite of inequality of contributions. This aligns with the suggestive evidence from various scientific fields where larger, heterogeneous research collaborations are linked to more differentiated, non-egalitarian credit attributions.
One aspect not considered here are the possibility of preferences over the distribution of outcomes. While not uncontested, there is suggestive evidence for inequality aversion and other distribution-related preferences. Since inequality of endowments requires inequality in rewards for cooperation to be compatible with individual incentives, such preferences might affect the relation between inequality, sharing, and cooperation. A possible extension of the work presented here might theoretically and experimentally investigate such interactions.

A Proofs

Proof for Lemma 1.

Proof. Suppose $\sum_i \frac{1}{r_i} \leq 1$. Consider the sharing rule $f_i = \frac{1}{r_i}$. Clearly, $f_i r_i = 1,\ \forall i \in N$. Furthermore, it is feasible since $\sum_i f_i = \sum_i \frac{1}{r_i} \leq 1$ by assumption.

Suppose now there is a feasible rule f_i such that $f_i r_i \geq 1,\ \forall i \in N$. Feasibility requires that $\sum_i f_i \leq 1$. But then $f_i \geq \frac{1}{r_i}$ and accordingly $\sum_i \frac{1}{r_i} \leq 1$ as required. □

Proof for Lemma 2.

Proof. Consider the following way of sharing:

$$\hat{f}_k = \frac{e_k}{\delta \sum_{j \neq k} e_j r_j + e_k r_k}, \quad \forall k \in N.$$

Condition SD can be written as:

$$\hat{f}_k r_k = \frac{e_k r_k}{\delta \sum_{j \neq k} e_j r_j + e_k r_k} < 1$$

As $\delta > 0$ and $e_j, r_j > 0,\ \forall j \in N$, this is satisfied for all i. By definition, $\hat{\mathbf{f}}$ satisfies IC, which is a necessary condition for full cooperation. It follows that if there is any feasible sharing rule that sustains cooperation in the corresponding Γ, then:

$$\sum_{k \in N} \frac{e_k}{\delta \sum_{j \neq k} e_j r_j + e_k r_k} \leq 1$$

which implies $\sum_{k \in N} \hat{f}_k \leq 1$. If $(\mathbf{e}, \mathbf{r}, \delta)$ allows for cooperation, then $\hat{\mathbf{f}} \in \mathcal{F}(\mathbf{e}, \mathbf{r}, \delta)$. □

Proof for Lemma 3.

Proof. Sufficiency: Suppose (4) holds. Setting $\hat{f}_k = \frac{e_k}{\delta \sum_{j \neq k} e_j r_j + e_k r_k}$ for all $k \in N$ yields a feasible sharing rule. By construction, (2) is satisfied. There is no incentive to deviate under strategy Grim. Furthermore, the punishment path for Grim is a Nash equilibrium. Thus full cooperation can be sustained as SPE in $\Gamma(\mathbf{e}, \mathbf{r}, \delta, \mathbf{f})$. According to Lemma 2, this is a social dilemma, as required by Definition 2.
Necessity: Suppose now a sharing rule $\mathbf{f}$ exists, that sustains full cooperation in equilibrium. Then it follows from (IC) that $f_k \geq \frac{e_k}{\delta \sum_{j \neq k} e_j r_j + e_k r_k},\ \forall k \in N$. But then there exists an $\hat{\mathbf{f}}$ with $\hat{f}_k = \frac{e_k}{\delta \sum_{j \neq k} e_j r_j + e_k r_k},\ \forall k \in N$ and necessarily $\hat{\mathbf{f}} \leq \mathbf{f}$. Accordingly, if $\mathbf{f}$ is feasible meaning $\sum_{k \in N} f_k \leq 1$ then so is $\hat{\mathbf{f}}$. Again, according to Lemma 2, the corresponding Γ is a social dilemma.

To prove the last statement, suppose the inequality (4) is strict. Then $\hat{\mathbf{f}} \in \mathcal{F}$, where $\hat{\mathbf{f}}$ is as defined previously. Furthermore, $\sum_{k\in N} \hat{f}_k < 1$. Let

$$\bar{\epsilon} \equiv \frac{1}{n} \cdot \Big(1 - \sum_{k\in N} \hat{f}_k\Big).$$

For any $\epsilon \in [0, \hat{\epsilon}]$, let $\mathbf{f}_\epsilon \equiv \hat{\mathbf{f}} + \epsilon$. Clearly, for any such $\mathbf{f}_\epsilon$, we have $\mathbf{f}_\epsilon \in \mathcal{F}$. Finally, if (4) holds with equality, then $\hat{f} \in \mathcal{F}$. As IC must also hold with equality, there can be no other element in $\mathcal{F}$. The result follows. □

Proof for Theorem 1.

Proof. Let $\bar{\mathbf{e}}$ be such that $\bar{e}_1 = 1$ while $\bar{e}_i = 0$, $\forall i \in \{2, ..., n\}$. Note that there is no social dilemma associated with $\bar{\mathbf{e}}$ that allows for cooperation since it would require $i = 1$ to have an incentive to invest in the joint project without the contribution of any other player. This would need $f_1 r_1 \geq 1$ meaning that there is no incentive to free ride and hence no 'dilemma'. Take any $\mathbf{e}$, $\mathbf{r}$, and δ and consider an infinite sequence $\{\mathbf{e}(1), \mathbf{e}(2), \mathbf{e}(3), ...\}$ with $\mathbf{e}(1) = \mathbf{e}$ that converges to $\bar{\mathbf{e}}$. There is an associated sequence of sharing rules $\{\hat{f}, \hat{f}(1), ...\}$, where $\hat{f}$ is as defined in the proof of Lemma 3. As $\{\mathbf{e}(k)\}_1^\infty$ is a convergent sequence, for any $\epsilon > 0$, there is a $K_\epsilon \in \mathbb{N}$ such that for all $k > K_\epsilon$, $\mathbf{e}(k)$ is such that $e_1(k) > 1 - \epsilon$. As $e_j > 0$, $\forall j \in N$ for any allowable endowment, we can conclude that $e_i(k) < \epsilon$, $\forall i \in \{2, ..., n\}$. This implies that

$$\hat{f}_1(k) \leq \frac{1}{\delta \epsilon \underline{r} + r_1}$$

where $\underline{r} = \min\{r_1, r_2, ..., r_n\}$. Furthermore, for all $j \in \{2, ..., n\}$:

$$\hat{f}_j(k) \leq \frac{\epsilon}{\delta(1-\epsilon)\underline{r} + \epsilon r_j}$$

Clearly, as $\epsilon \to 0$, $\hat{f}_j \to 0$ for any $j \neq 1$ while $\hat{f}_1 \to \frac{1}{r_1}$. But as $r_1 > 1$ by assumption, $\sum_i \hat{f}_i \to \frac{1}{r_1} < 1$. As $\hat{f}_i$ is a continuous function of $\mathbf{e}$, the sequence $\{\hat{f}, \hat{f}(1), ...\}$ converges to $\overline{\mathbf{f}}^*$ which is such that $\overline{f}_1^* = \frac{1}{r_1}$ and $\overline{f}_j^* = 0$ for all $j \neq 1$. Let $\phi = 1 - \frac{1}{r_1}$. There exists K_ϕ such that for all $k > K_\phi$, $\sum_i \hat{f}_i(k) - \frac{1}{r_1} < \phi$ and thus $\sum_i \hat{f}_i(k) < 1$. Let ϵ^* be such that

$$\frac{1}{\delta \epsilon^* \underline{r} + r_1} + \frac{(n-1)\epsilon^*}{\delta(1-\epsilon^*)\underline{r} + \epsilon^* r_j} = 1.$$

It follows that for any $k > K_{\epsilon^*}$ where K_{ϵ^*} is defined as before, $e_1(k) > 1 - \epsilon^*$ and thus $\sum_i \hat{f}_i(k) < 1$, which implies $k > K_\phi$. For any such $\mathbf{e}(k)$, $(\mathbf{e}(k), \mathbf{r}, \delta)$ allows for cooperation. □

Proof for Proposition 1.

Proof. First, we show that the constraint that determines whether the optimal shares are feasible reaches an extremum point at the equal endowment distribution. Recall that feasibility is satisfied if $\sum_i f_i \leq 1$.

Writing this out we get:

$$\sum_{j\in N} f_j = \frac{e_i}{\delta\sum_{j\neq i} e_j r_j + e_i r_i} + \sum_{j\neq i} \frac{e_j}{\delta\sum_{k\neq j} e_k r_k + e_j r_j}$$

Taking the partial derivative w.r.t. e_i:

$$\frac{\partial}{\partial f_i}\sum_{j\in N} f_j = \frac{\delta\sum_{j\neq i} e_j r_j}{\left(\delta\sum_{j\neq i} e_j r_j + e_i r_i\right)^2} - \sum_{j\neq i} \frac{\delta e_j r_i}{\left(\delta\sum_{k\neq j} e_k r_k + e_j r_j\right)^2}$$

Let $e_i = e_j = e$ and $r_i = r_j = r$:

$$\frac{\partial}{\partial f_i}\sum_{j\in N} f_j = \frac{\delta e(n-1)r}{\left(\delta(n-1)er + er\right)^2} - \sum_{j\neq i} \frac{\delta er}{\left(\delta(n-1)er + er\right)^2} \equiv 0$$

It follows from Lemma 5 that this is a global maximum. This implies that if there is a sharing rule that sustains full cooperation as SPE at equal endowments, we can find a feasible sharing rule that sustains full cooperation for any endowment distribution. Furthermore, as $\frac{\partial f_i}{\partial r_j} < 0,\ \forall i,j$ and $\frac{\partial f_i}{\partial \delta} < 0,\ \forall i$, increasing any r_i or δ relaxes the constraint. This means that a sharing rule that sustains full cooperation as SPE for a given endowment also sustains this for any $\hat{\delta} > \delta$ or any productivity vector $\hat{\mathbf{r}} \geq \mathbf{r}$ where the vectors are compared element wise. □

Proof for Proposition 2.

Proof. Following Lemma 3, to determine whether any $(\mathbf{e}, \mathbf{r}, \delta)$ allows for cooperation, it suffices to check if $\sum_i \hat{f}_i \leq 1$, where $\hat{f}$ is as defined in the proof of Lemma 3.

Case 1: equal productivities

It follows from Lemma 5 that for equal productivities, the minimum shares $\hat{\mathbf{f}}$ that need to be guaranteed to each individual for no deviation to be profitable are strictly concave in $\mathbf{e}$. This means if $\sum_i \hat{f}_i(\mathbf{e}) = c \leq 1$, then we can find $\hat{e}$ such that $\sum_i \hat{f}_i(\hat{\mathbf{e}}) < c$. Due to continuity of $\hat{f}$ in $\mathbf{r}$ and δ, we can find $\hat{\mathbf{r}} < \mathbf{r}$ and $\hat{\delta} < \delta$ such that still $\sum_i \hat{f}_i^*(\hat{\mathbf{e}}) < 1$ meaning that a sharing rule implementing full cooperation as SPE for these parameters exists.

Case 2: unequal productivities
Take $i, j \in N$ such that $e_i r_i \geq e_j r_j$ and $r_i \neq r_j$. As productivities are unequal, such i, j necessarily exist. Furthermore, denote with $\hat{e}$ a new endowment allocation that only differs from $\mathbf{e}$ in i and j meaning that $\hat{e}_k = e_k$ for all $k \neq i, j$. As $\sum_j e_j = 1$ for any $\mathbf{e}$, this implies that $\hat{e}_i \equiv e_i + (e_j - \hat{e}_j)$.

Case 2.a: $r_i \geq r_j$

Take some $\hat{e}_i > e_i$. Full cooperation at this endowment can be feasibly implemented for δ and $\mathbf{r}$ if $\sum_i \hat{f}_i(\hat{e}) < \sum_i \hat{f}_i(e)$. This can be written out as:

$$\sum_{k \neq i,j} \frac{e_k}{\delta \sum_{j \in N} \hat{e}_j r_j + (1-\delta) e_k r_k} + \frac{\hat{e}_i}{\delta \sum_{j \in N} \hat{e}_j r_j + (1-\delta) \hat{e}_i r_i} + \frac{\hat{e}_j}{\delta \sum_{j \in N} \hat{e}_j r_j + (1-\delta) \hat{e}_j r_j}$$
$$< \sum_{k \neq i,j} \frac{e_k}{\delta \sum_{j \in N} e_j r_j + (1-\delta) e_k r_k} + \frac{e_i}{\delta \sum_{j \in N} e_j r_j + (1-\delta) e_i r_i} + \frac{e_j}{\delta \sum_{j \in N} e_j r_j + (1-\delta) e_j r_j}$$

As by assumption $r_i \geq r_j$, $e_i r_i \geq e_j r_j$ and by construction $\hat{e}_i > e_i$, it follows $\sum_{j \in N} \hat{e}_j r_j > \sum_{j \in N} e_j r_j$ and $\hat{e}_i r_i > e_i r_i$. Define $\delta \sum_{j \in N} e_j r_j \equiv A$. The above inequality holds if the following holds:

$$\sum_{k \neq i,j} \frac{e_k}{A + (1-\delta) e_k r_k} + \frac{\hat{e}_i}{A + (1-\delta) \hat{e}_i r_i} + \frac{\hat{e}_j}{A + (1-\delta) \hat{e}_j r_j}$$
$$\leq \sum_{k \neq i,j} \frac{e_k}{A + (1-\delta) e_k r_k} + \frac{e_i}{A + (1-\delta) e_i r_i} + \frac{e_j}{A + (1-\delta) e_j r_j}$$

which reduces to:

$$\frac{\hat{e}_i}{A + (1-\delta) \hat{e}_i r_i} + \frac{\hat{e}_j}{A + (1-\delta) \hat{e}_j r_j} \leq \frac{e_i}{A + (1-\delta) e_i r_i} + \frac{e_j}{A + (1-\delta) e_j r_j}$$

or equivalently:

$$\frac{\hat{e}_i}{A + (1-\delta) \hat{e}_i r_i} - \frac{e_i}{A + (1-\delta) e_i r_i} \leq \frac{e_j}{A + (1-\delta) e_j r_j} - \frac{\hat{e}_j}{A + (1-\delta) \hat{e}_j r_j}$$

This can be re-arranged to:

$$\frac{A \cdot \Delta_{ij}}{\big(A + (1-\delta) \hat{e}_i r_i\big)\big(A + (1-\delta) e_i r_i\big)} \leq \frac{A \cdot \Delta_{ij}}{\big(A + (1-\delta) e_j r_j\big)\big(A + (1-\delta) \hat{e}_j r_j\big)}$$

where $\Delta_{ij} \equiv \hat{e}_i - e_i = e_j - \hat{e}_j$. Noting that all factors are positive, this is satisfied if:

$$\big(A + (1-\delta) \hat{e}_i r_i\big)\big(A + (1-\delta) e_i r_i\big) \geq \big(A + (1-\delta) e_j r_j\big)\big(A + (1-\delta) \hat{e}_j r_j\big)$$

As by construction $\hat{e}_i r_i > e_i r_i \geq e_j r_j > \hat{e}_j r_j$, this is satisfied.

Case 2.b: $r_i < r_j$
As $e_i r_i \geq e_j r_j$ by assumption, we can conclude $e_i > e_j$. Set $\hat{e}_i = e_j$ and $\hat{e}_j = e_i$. Full cooperation can be sustained at $\hat{e}$ if:

$$\sum_{k\neq i,j}\frac{e_k}{\delta\sum_{j\in N}\hat{e}_j r_j+(1-\delta)e_k r_k}+\frac{\hat{e}_i}{\delta\sum_{j\in N}\hat{e}_j r_j+(1-\delta)\hat{e}_i r_i}+\frac{\hat{e}_j}{\delta\sum_{j\in N}\hat{e}_j r_j+(1-\delta)\hat{e}_j r_j}$$
$$<\sum_{k\neq i,j}\frac{e_k}{\delta\sum_{j\in N} e_j r_j+(1-\delta)e_k r_k}+\frac{e_i}{\delta\sum_{j\in N} e_j r_j+(1-\delta)e_i r_i}+\frac{e_j}{\delta\sum_{j\in N} e_j r_j+(1-\delta)e_j r_j}$$

As $\hat{e}_j = e_i > e_j = \hat{e}_i$ and $r_j > r_i$ which implies $\sum_{j\in N}\hat{e}_j r_j > \sum_{j\in N} e_j r_j$, the previous inequality satisfied if the following is satisfied:

$$\frac{\hat{e}_i}{A+(1-\delta)\hat{e}_i r_i}+\frac{\hat{e}_j}{A+(1-\delta)\hat{e}_j r_j}\leq\frac{e_i}{A+(1-\delta)e_i r_i}+\frac{e_j}{A+(1-\delta)e_j r_j}$$

where $A \equiv \delta\sum_{j\in N} e_j r_j$. Substituting in $\hat{e}_i = e_j$ and $\hat{e}_j = e_i$, we get:

$$\frac{e_j}{A+(1-\delta)e_j r_i}-\frac{e_j}{A+(1-\delta)e_j r_j}\leq\frac{e_i}{A+(1-\delta)e_i r_i}-\frac{e_i}{A+(1-\delta)e_i r_j}$$

or equivalently:

$$\frac{(1-\delta)e_j^2(r_j-r_i)}{(A+(1-\delta)e_j r_i)(A+(1-\delta)e_j r_j)}\leq\frac{(1-\delta)e_i^2(r_j-r_i)}{(A+(1-\delta)e_i r_i)(A+(1-\delta)e_i r_j)}$$

As $r_j > r_i$ and $e_j < e_i$, this is satisfied if the following is satisfied:

$$\frac{\big(A+(1-\delta)e_j r_i\big)\big(A+(1-\delta)e_j r_j\big)}{e_j}\geq\frac{\big(A+(1-\delta)e_i r_i\big)\big(A+(1-\delta)e_i r_j\big)}{e_i}$$

or equivalently:

$$\big(Ae_i+(1-\delta)e_i e_j r_i\big)\big(Ae_i+(1-\delta)e_i e_j r_j\big)\geq\big(Ae_j+(1-\delta)e_i e_j r_i\big)\big(Ae_j+(1-\delta)e_i e_j r_j\big)$$

Comparing terms, this holds if $Ae_i > Ae_j$, which is satisfied as $e_i > e_j$ by assumption.

□

Lemma 5. *The sharing rule* $\hat{\mathbf{f}}$ *with*

$$\hat{f}_k=\frac{e_k}{\delta\sum_{j\neq k} e_j r_j+e_k r_k},\quad \forall k\in N.$$

for some $\mathbf{e}$, $\mathbf{r}$, *and* δ *is such that* $\sum_i \hat{f}_i$ *is*

(i) strictly concave in e_k *for any* r_k *with* $r_k = r,\ \forall k$;

(ii) strictly convex in r_k *for any* e_k *with* $e_k = e,\ \forall k$.

Proof. (i) Note that the sharing rule for a fixed r can be written as

$$f(\mathbf{e}) := \sum_{k=1}^{n} f_k(\mathbf{e}) = \sum_{k=1}^{n} \frac{e_k}{\delta \sum_{j \neq k} e_j r + e_k r} \tag{11}$$

By the definition of strict concavity, the following should hold

$$f(\lambda \mathbf{e} + (1-\lambda)\tilde{\mathbf{e}}) > \lambda f(\mathbf{e}) + (1-\lambda) f(\tilde{\mathbf{e}}),\ \lambda \geq 0$$

In addition, a sum of concave functions is a concave function. Hence, it is sufficient to show concavity of only one function $f_k(\mathbf{e})$ for some k. First note that

$$f_k(\lambda \mathbf{e} + (1-\lambda)\tilde{\mathbf{e}}) = \frac{\lambda e_k + (1-\lambda)\tilde{e}_k}{\delta \sum_{j \neq k} (\lambda e_j + (1-\lambda)\tilde{e}_j) r + (\lambda e_k + (1-\lambda)\tilde{e}_k) r}$$

and

$$\lambda f_k(\mathbf{e}) + (1-\lambda) f_k(\tilde{\mathbf{e}}) = \frac{\lambda e_k}{\delta \sum_{j \neq k} e_j r + (1-\delta) e_k r} + \frac{(1-\lambda)\tilde{e}_k}{\delta \sum_{j \neq k} \tilde{e}_j r + (1-\delta)\tilde{e}_k r}$$

Hence, we need that the following inequality holds

$$\frac{\lambda e_k + (1-\lambda)\tilde{e}_k}{\delta \sum_{j \neq k} (\lambda e_j + (1-\lambda)\tilde{e}_j) + (1-\delta)(\lambda e_k + (1-\lambda)\tilde{e}_k)} > \frac{\lambda e_k}{\delta \sum_{j \neq k} e_j + (1-\delta) e_k} + \frac{(1-\lambda)\tilde{e}_k}{\delta \sum_{j \neq k} \tilde{e}_j + (1-\delta)\tilde{e}_k}$$

Since $\sum_{i=1}^{n} e_i = 1$, the inequality can be reduced to

$$\frac{\lambda e_k + (1-\lambda)\tilde{e}_k}{\lambda(\delta + (1-\delta) e_k) + (1-\lambda)(\delta + (1-\delta)\tilde{e}_k)} > \frac{\lambda e_k}{\delta + (1-\delta) e_k} + \frac{(1-\lambda)\tilde{e}_k}{\delta + (1-\delta)\tilde{e}_k}$$

Let us introduce the following substitution:

$$\begin{aligned} a &:= \lambda e_k \\ b &:= (1-\lambda)\tilde{e}_k \\ c &:= \delta + (1-\delta) e_k \\ d &:= \delta + (1-\delta)\tilde{e}_k \end{aligned} \tag{12}$$

Then, we can re-arrange the inequality to obtain

$$\begin{aligned} (a+b)cd &> (ad+bc)(\lambda c + (1-\lambda) d) \Rightarrow \\ acd + bcd &> \lambda acd + (1-\lambda) ad^2 + \lambda bc^2 + (1-\lambda) bcd \Rightarrow \\ (1-\lambda) ad(c-d) &> \lambda bc(c-d) \end{aligned}$$

Hence, we need to consider two cases: when $c > d$ and $c < d$. If $c = d$, then the inequality is satisfied and the statement follows. If $c > d$, then $e_k > \tilde{e}_k$ and

$$(1 - \lambda)ad > \lambda bc$$

Using the substitution (12), we obtain

$$\frac{\lambda e_k}{\delta + (1 - \delta)e_k} > \frac{(1 - \lambda)\tilde{e}_k}{\delta + (1 - \delta)\tilde{e}_k}$$

which is satisfied whenever $e_k > \tilde{e}_k$. The same argument works for the second case, which completes the proof of part (i).

(ii) Next, note that the sharing rule for a fixed e can be written as

$$f(\mathbf{r}) := \sum_{k=1}^{n} f_k(\mathbf{r}) = \sum_{k=1}^{n} \frac{e}{\delta \sum_{j \neq k} er_j + (1 - \delta)er_k} \tag{13}$$

By the definition of strict convexity, the following should hold

$$f(\lambda \mathbf{r} + (1 - \lambda)\tilde{\mathbf{r}}) < \lambda f(\mathbf{r}) + (1 - \lambda) f(\tilde{\mathbf{r}}),\ \lambda \geq 0$$

As in (i), it is sufficient to show convexity of only one function $f_k(\mathbf{r})$ for some k. First note that

$$f_k(\lambda \mathbf{r} + (1 - \lambda)\tilde{\mathbf{r}}) = \frac{1}{\delta \sum(\lambda r_j + (1 - \lambda)\tilde{r}_j) + (1 - \delta)(\lambda r_k + (1 - \lambda)\tilde{r}_k)}$$

and

$$\lambda f_k(\mathbf{r}) + (1 - \lambda) f_k(\tilde{\mathbf{r}}) = \frac{\lambda}{\delta \sum r_j + (1 - \delta)r_k} + \frac{1 - \lambda}{\delta \sum \tilde{r}_j + (1 - \delta)\tilde{r}_k}$$

After re-arranging the terms in the inequality in the similar manner as in (i) and letting

$$a := \delta \sum r_i + (1 - \delta)r_k$$
$$b := \delta \sum \tilde{r}_i + (1 - \delta)\tilde{r}_k$$

we obtain

$$ab < (\lambda b + (1 - \lambda)a)(\lambda a + (1 - \lambda b) \Rightarrow$$
$$2ab < (a^2 + b^2) \Rightarrow$$
$$(a - b)^2 < 0$$

which completes the proof. □

Proof for Theorem 2.

Proof. Suppose first that productivities $\mathbf{r}$ are equal across players. Take any endowment $\mathbf{e}$ and suppose without loss of generality that $\mathbf{e}$ is ordered such that $e_i \leq e_{i+1}$. Construct an $\dot{\mathbf{e}}$ as follows: $\hat{e}_1 = e_1 - \kappa$ for some $\kappa \in (0, e_1)$, $\dot{e}_n = e_n + \kappa$, and $\dot{e}_i = e_i$ for all other $i \in N$. Construct a second $\ddot{\mathbf{e}}$ with $\ddot{e}_1 = \dot{e}_n$, $\ddot{e}_n = \dot{e}_1$, and $\ddot{e}_i = \dot{e}_i$ for all other $i \in N$. Clearly, endowments $\dot{\mathbf{e}}$ and $\ddot{\mathbf{e}}$ are more unequal than $\mathbf{e}$. As productivities are equal, full cooperation achieves the same aggregate return from investment in the joint project for all these endowments. Furthermore, define the sharing rule $\dot{\mathbf{f}}$ as follows:

$$\dot{f}_k = \frac{\dot{e}_k}{\delta \sum_{j \neq k} \dot{e}_j r_j + \dot{e}_k r_k}, \quad \forall k \in N.$$

Define $\ddot{\mathbf{f}}$ equivalently. Furthermore, let $\hat{\mathbf{f}}$ be the equivalent rule for $\mathbf{e}$. Clearly, all three satisfy IC. Furthermore, $\dot{\mathbf{e}}$ and $\ddot{\mathbf{e}}$ are identical except that $\dot{f}_1$ and $\dot{f}_n$ are interchanged for $\ddot{\mathbf{f}}$. This implies that $\sum_i \dot{f}_i = \sum_i \ddot{f}_i$. According to Lemma 5, $\sum_i \hat{f}_i$ is strictly concave in $\mathbf{e}$. It follows from strict concavity that since $\mathbf{e}$ can be written as a convex combination of $\dot{\mathbf{e}}$ and $\ddot{\mathbf{e}}$, and since $\sum_i \dot{f}_i = \sum_i \ddot{f}_i$, that $\sum_i \hat{f}_i > \sum_i \dot{f}_i$. As whether or not any sharing rule $\mathbf{f}$ can be implemented depends on whether $\sum_i f_i \leq 1$, and as $\hat{f}_i$ is increasing and continuous in δ, we can conclude that if for some δ, $\sum_i \hat{f}_i = 1$, then for the same δ, we have $\sum_i \dot{f}_i < 1$. This means there exists a $\dot{\delta} < \delta$ such that $(\dot{\mathbf{e}}, \mathbf{r}, \dot{\delta})$ allows for cooperation but $(\mathbf{e}, \mathbf{r}, \dot{\delta})$ does not. For $\mathbf{r}$, cooperation is easier to sustain with $\dot{\mathbf{e}}$ than $\mathbf{e}$, as asserted.

Suppose now that productivities $\mathbf{r}$ are not equal across players. Let again endowment $\mathbf{e}$ be ordered such that $e_i \leq e_{i+1}$. If $r_n \geq r_1$, it follows immediately from the proof of Proposition 2 (Case 2.a) that setting $\dot{e}_1 = e_1 - \kappa$ for some $\kappa \in (0, e_1)$ and $\dot{e}_n = e_n + \kappa$, while leaving all other endowments unchanged, means that for every δ, the corresponding $\sum_i \dot{f}_i < \sum_i \hat{f}_i$. Following the same argument as above, we can thus find a $\hat{\delta}$ such that $(\dot{\mathbf{e}}, \mathbf{r}, \dot{\delta})$ allows for cooperation but $(\mathbf{e}, \mathbf{r}, \dot{\delta})$ does not.
If instead If $r_n < r_1$, we can construct an endowment $\dot{\mathbf{e}}$ as follows: $\dot{e}_n = e_1 - \kappa$ for some $\kappa \in (0, e_1)$ and $\dot{e}_1 = e_n + \kappa$, and $e_i = \dot{e}_i$ otherwise. Again, $\dot{\mathbf{e}}$ is more unequal than $\mathbf{e}$. It follows from from the proof of Proposition 2 (Case 2.b) that we can find a $\dot{\delta}$ such that $(\dot{\mathbf{e}}, \mathbf{r}, \dot{\delta})$ allows for cooperation but $(\mathbf{e}, \mathbf{r}, \dot{\delta})$ does not. The result follows. □

Proof for Proposition 3.

Proof. The proof of Theorem 1 already derived such an ϵ for an (arbitrarily chosen) player 1. We can apply the same argument to obtain an ϵ, which - given $\mathbf{r}$ and δ - applies to all i not just $i = 1$.
Recall that the proof of Theorem 1 considered sequences converging to $\overline{\mathbf{e}}$ where $\overline{e}_1 = 1$ and $\overline{e}_{i \neq 1} = 0$. We can construct such a limiting endowment for all i: Let $\overline{\mathbf{e}}(i)$ be the endowment vector where $\overline{e}_i = 1$. For each of these, we get some K_{ϵ^*} which we denote by $K_{\epsilon^*}(i)$. Then we can define $\epsilon \equiv \min_{i \in N}\{K_{\epsilon^*}(i)\}$. Given $\mathbf{r}$ and δ, if for any endowment $\mathbf{e}$ we have $e_j > 1 - \epsilon$ for some $j \in N$, then by definition the

corresponding shares are such that $\sum_i \hat{f}_i < 1$ and cooperation can be sustained in equilibrium. This implies any $(\mathbf{e}, \mathbf{r}, \delta)$ with $e_i > 1 - \epsilon$, for some i, allows for cooperation. □

Proof for Lemma 4.

Proof. According to Assumption 1, $\sum_i r_i^{-1} > 1$. As $(\mathbf{e}, \mathbf{r}, \delta)$ allows for cooperation, $\hat{\mathbf{f}}$ sustains full cooperation. From the definition of $\hat{\mathbf{f}}$, we can see that for all i, $\hat{f}_i$ is strictly increasing in δ. Therefore, if $\hat{\mathbf{f}}$ can sustain cooperation in equilibrium, so can $\mathbf{f}^*$ as by definition $\delta_{\text{min}} \leq \delta$. Furthermore, if $\delta_{\text{min}} > 0$, then $\sum_i {f_i}^* = 1$. If not and instead $\sum_i {f_i}^* < 1$, it follows from continuity and monotonicity in δ that we could find a $\delta < \delta_{\text{min}}$ that would still allow for cooperation, which contradicts the definition of δ_{min}. For $\delta_{\text{min}} > 0$, it must be that at $\delta = 0$:

$$\sum_{i \in N} \frac{e_i}{\delta \sum_{j \neq i} e_j r_j + e_i r_i} = \sum_{i \in N} \frac{e_i}{e_i r_i} = \sum_{i \in N} \frac{1}{r_i} > 1$$

which is satisfied by assumption. □

Proof for Proposition 5.

Proof. It follows from Lemma 3 and Lemma 4 that if cooperation can be implemented with $\mathbf{f}^*$ it can also be implemented with the corresponding $\hat{\mathbf{f}}$. Full cooperation under equal sharing can be implemented as long as the minimum share that needs to be awarded to each agent does not exceed $\frac{1}{n}$. Formally, we require:

$$\max\Big\{\hat{f}_i\Big\}_{i \in N} \equiv \max\Big\{\frac{e_i}{\delta \sum_{j \neq i} e_j r_j + e_i r_i}\Big\}_{i \in N} \leq \frac{1}{n}$$

For generic $\mathbf{e}$ and $\mathbf{r}$, $\frac{e_i}{\delta \sum_{j \neq i} e_j r_j + e_i r_i} \neq \frac{e_k}{\delta \sum_{j \neq i} e_j r_j + e_k r_k}$ for $k \neq i$. For those parameters, there exist some $i, k \in N$ such that

$$\frac{e_i}{\delta \sum_{j \neq i} e_j r_j + e_i r_i} > \frac{e_k}{\delta \sum_{j \neq i} e_j r_j + e_k r_k}$$

And thus $\sum_{j \in N} \hat{f}_j < n\hat{f}_i \leq 1$. As $\hat{f}_i$ is continuous and decreasing in δ, we can find $\tilde{\delta} \leq \delta$ such that

$$\max\Big\{\tilde{f}_j\Big\}_{j \in N} = \tilde{f}_i^* = \frac{1}{n}.$$

where the first equality follows from the fact that δ scales each share $\hat{f}_i$ in an order-preserving way, meaning that for any $\tilde{\delta} < \delta$, player i still needs to be awarded the largest share. Under equal sharing, full cooperation is (just) feasible for these parameters, but not feasible for any $\delta < \underline{\delta}$. But as generically $\sum_{j \in N} \hat{f}_j < n\hat{f}_i$, it follows that $\sum_{j \in N} \tilde{f}_j^* < n\tilde{f}^* = 1$ and thus $\tilde{\delta} > \delta_{\text{min}}$. This implies that full cooperation can be implemented under $\mathbf{f}^*$. Due to continuity we can find $\tilde{r}_j < \underline{r}_j$ (and $\delta < \tilde{\delta}$ for which full cooperation can still be implemented as SPE with the sharing rule $\mathbf{f}^*$ while this is not the case for equal sharing. □

Proof for Proposition 4.

Proof. Note that equation (5) can be re-written as

$$\underbrace{\delta_{\min}\Big(f_i^*\sum_{k\neq i} e_k r_k\Big)}_{\text{cost of punishment for defection}} - \overbrace{\Big(e_i - f_i^* e_i r_i\Big)}^{\text{private benefit from defection}} = 0$$

We can further re-arrange these conditions as

$$\frac{f_i^*(e_i - e_i r_i) + e_i \sum_{j\neq i} f_j^*}{\sum_{k\neq i} e_k r_k} = \delta_{\min} f_i^*$$

Hence, accounting for all players, we obtain a system of n linear equations given by

$$\Phi \mathbf{f}^* = \delta_{\min}\mathbf{f}^*$$

where matrix Φ is such that

$$\phi_{ij} = \begin{cases} \dfrac{-e_i(r_i - 1)}{\sum_{k\neq i} e_k r_k}, & i = j \\ \dfrac{e_i}{\sum_{k\neq i} e_k r_k}, & i \neq j \end{cases}$$

Thus, $\delta_{\min}$ is an eigenvalue of Φ and $\mathbf{f}^*$ is the corresponding eigenvector such that $\sum f_i^* = 1$. Moreover, $\delta_{\min}$ is the largest eigenvalue of Φ. Matrix Φ can be interpreted as the matrix of the cost of cooperation weighted by contributions of others. Since it follows from our results that for given $\mathbf{e}$ and $\mathbf{r}$ we can always find δ such that cooperation can be sustained, then matrix Φ always has an eigenvalue $\delta_{\min} \in (0, 1)$.

□

Proof for Proposition 6.

Proof. From

$$\Phi \mathbf{f}^* = \delta_{\min}\mathbf{f}^*$$

we see that equal sharing is an optimal sharing rule if and only if Φ is a row-sum-constant matrix. Then, we require

$$-e_i(r_i - 1) + (n-1)e_i - \delta_{\min} e_k r_k = \delta_{\min} \sum_{j\neq i,k} e_j r_j, \ \forall i$$

Hence, for any two players i and k the following has to be true

$$-e_i(r_i - 1) + (n-1)e_i - \delta_{\min} e_k r_k = -e_k(r_k - 1) + (n-1)e_k - \delta_{\min} e_i r_i$$

which can be re-arranged as

$$\frac{e_i}{e_k} = \frac{n - (1 - \delta_{\min}) r_i}{n - (1 - \delta_{\min}) r_k},\ \forall i, k \tag{14}$$

If $r_i = r_k$, then $e_i = e_k$ for all players. For $r_i \neq r_k$, since for any given δ there is a unique endowment distribution satisfying

$$\frac{e_i}{e_k} = \frac{n - (1 - \delta) r_i}{n - (1 - \delta) r_k},\ \forall i, k \tag{15}$$

we can conclude that the endowment distribution satisfying (14) is not generic. □

Proof for Proposition 7.

Proof. For ${f_k}^*$ to be equal to g_k^* we need

$$\delta_{\min} \sum_{j \neq k} e_j r_j + e_k r_k = 1. \tag{16}$$

If $r_k = r,\ \forall k$, then this condition can be re-written as

$$e_k = \frac{1 - \delta_{\min} r}{r - \delta_{\min} r},\ \forall k,$$

which implies that $e_k = 1/n$.

Assume now that productivities of players differ. Assume also that for some k and i, condition (16) is satisfied. Then,

$$\delta_{\min} \sum_{j \neq i,k} e_j r_j + \delta_{\min} e_k r_k = 1 - e_i r_i$$
$$\delta_{\min} \sum_{j \neq i,k} e_j r_j + \delta_{\min} e_i r_i = 1 - e_k r_k$$

and hence

$$\frac{e_i}{e_k} = \frac{r_k}{r_i}.$$

Hence, for any given productivities of players, there is a unique endowment distribution for which $g_k^* = {f_k}^*$ for all k.

□

Proof for Proposition 8.

Proof. If $r_k = r,\ \forall k$, then ${f_k}^* = h_k^*$ if

$$e_k = \frac{1 - \delta_{\min} r}{r - \delta_{\min} r},\ \forall k,$$

which implies that $e_k = {}^1\!/_n$.
Now consider $r_i \neq r_k$ for at least some i, k. First, consider the sharing rule $\hat{f}_k$ for some δ, then we need to compare

$$h_k^* = \frac{e_k r_k}{\sum_{j \neq k} e'_j r_j + e_k r_k}$$
$$f_k{}^* = \frac{e_k}{\delta \sum_{j \neq k} e'_j r_j + e_k r_k}$$

Assume that these rules are equal for some k. Then, for $h_k^* = f_k{}^*$ for all k, we need that

$$\frac{e_k r_k}{e_i r_i} = \frac{(1 - \delta r_k) r_i}{(1 - \delta r_i) r_k}$$

If the shares are equal, then either change in productivities or endowment re-distribution has to have to same effect on these rules. That is, if we increase the productivity for player k, then the following equation has to hold

$$\frac{e_k (r_k + \epsilon)}{e_i r_i} = \frac{(1 - \delta(r_k + \epsilon)) r_i}{(1 - \delta r_i)(r_k + \epsilon)},$$

and while the left-hand side of the equation is increased, the right-hand side is necessarily decreased, which is a contradiction. Similarly, assume we increase the endowment of player k. For the endowment re-distribution to have the same effect on both rules we need

$$\frac{(e_k + \epsilon_k) r_k}{(e_i - \epsilon_i) r_i} = \frac{(1 - \delta r_k) r_i}{(1 - \delta r_i) r_k} = \frac{e_k r_k}{e_i r_i},$$

which can be re-written as

$$\frac{(e_k + \epsilon_k)}{(e_i - \epsilon_i)} = \frac{e_k}{e_i}$$

and thus

$$\epsilon_k e_i = -\epsilon_i e_k,$$

which is a contradiction. This indicates that for a given set of productivities and δ, there exists a unique endowment distribution such that $h_k^* = f_k{}^*, \ \forall k$ defined as

$$\frac{e_k r_k}{e_i r_i} = \frac{(1 - \delta r_k) r_i}{(1 - \delta r_i) r_k}$$

By continuity of $\delta_{\min}$, there exists only one endowment distribution $\hat{\mathbf{e}}$ such that $h_k^* = f_k{}^*, \ \forall k$ defined as

$$\frac{\hat{e}_k r_k}{\hat{e}_i r_i} = \frac{(1 - \delta_{\min} r_k) r_i}{(1 - \delta_{\min} r_i) r_k}$$

□

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